6 Homotopy invariance of homology

We now know that the homology of a star-shaped region is trivial: in such a space, every cycle with augmentation 0 is a boundary. We will use that fact, which is a special case of homotopy invariance of homology, to prove the general result, which we state in somewhat stronger form:

Theorem 6.1. *A homotopy $h : f_0 \simeq f_1 : X \to Y$ determines a natural chain homotopy $f_{0*} \simeq f_{1*} : S_*(X) \to S_*(Y)$.*

The proof uses naturality (a lot). For a start, notice that if $k : g_0 \simeq g_1 : C_* \to D_*$ is a chain homotopy, and $j : D_* \to E_*$ is another chain map, then the composites $j \circ k_n : C_n \to E_{n+1}$ give a chain homootpy $j \circ g_0 \simeq j \circ g_1$. So if we can produce a chain homotopy between the chain maps induced by the two inclusions $i_0, i_1 : X \to X \times I$, we can get a chain homotopy k between $f_{0*} = h_* \circ i_{0*}$ and $f_{1*} = h_* \circ i_{1*}$ in the form $h_* \circ k$.

So now we want to produce a natural chain homotopy, with components $k_n : S_n(X) \to S_{n+1}(X \times I)$. The unit interval hosts a natural 1-simplex given by an identification $\Delta^1 \to I$, and we should imagine k as being given by "multiplying" by that 1-chain. This "multiplication" is a special case of a chain map

$$\times : S_*(X) \times S_*(Y) \to S_*(X \times Y)\,,$$

defined for any two spaces X and Y, with lots of good properties. It will ultimately be used to compute the homology of a product of two spaces in terms of the homology groups of the factors.

Here's the general result.

Theorem 6.2. *There exists a map $\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$, the* cross product*, that is:*

- *Natural, in the sense that if $f : X \to X'$ and $g : Y \to Y'$, and $a \in S_p(X)$ and $b \in S_p(Y)$ so that $a \times b \in S_{p+q}(X \times Y)$, then $f_*(a) \times g_*(b) = (f \times g)_*(a \times b)$.*
- *Bilinear, in the sense that $(a + a') \times b = (a \times b) + (a' \times b)$, and $a \times (b + b') = a \times b + a \times b'$.*
- *The Leibniz rule is satisfied, i.e., $d(a \times b) = (da) \times b + (-1)^p a \times db$.*
- *Normalized, in the following sense. Let $x \in X$ and $y \in Y$. Write $j_x : Y \to X \times Y$ for $y \mapsto (x, y)$, and write $i_y : X \to X \times Y$ for $x \mapsto (x, y)$. If $b \in S_q(Y)$, then $c^0_x \times b = (j_x)_* b \in S_q(X \times Y)$, and if $a \in S_p(X)$, then $a \times c^0_y = (i_y)_* a \in S_p(X \times Y)$.*

The Leibniz rule contains the first occurence of the "topologist's sign rule"; we'll see these signs appearing often. Watch for when it appears in our proof.

Proof. We're going to use induction on $p+q$; the normalization axiom gives us the cases $p+q = 0, 1$. Let's assume that we've constructed the cross-product in total dimension $p + q - 1$. We want to define $\sigma \times \tau$ for $\sigma \in S_p(X)$ and $\tau \in S_q(Y)$.

Note that there's a universal example of a p-simplex, namely the identity map $\iota_p : \Delta^p \to \Delta^p$. It's universal in the sense any p-simplex $\sigma : \Delta^p \to X$ can be written as $\sigma_*(\iota_p)$ where $\sigma_* : \mathrm{Sin}_p(\Delta^p) \to \mathrm{Sin}_p(X)$ is the map induced by σ. To define $\sigma \times \tau$ in general, then, it suffices to define $\iota_p \times \iota_q \in S_{p+q}(\Delta^p \times \Delta^q)$; we can (and must) then take $\sigma \times \tau = (\sigma \times \tau)_*(\iota_p \times \iota_q)$.

Our long list of axioms is useful in the induction. For one thing, if $p = 0$ or $q = 0$, normalization provides us with a choice. So now assume that both p and q are positive. We want the cross-product to satisfy the Leibnitz rule:

$$d(\iota_p \times \iota_q) = (d\iota_p) \times \iota_q + (-1)^p \iota_p \times d\iota_q \in S_{p+q-1}(\Delta^p \times \Delta^q)$$

Since $d^2 = 0$, a necessary condition for $\iota_p \times \iota_q$ to exist is that $d((d\iota_p) \times \iota_q + (-1)^p \iota_p \times d\iota_q) = 0$. Let's compute what this is, using the Leibniz rule in dimension $p + q - 1$ where we have it by the inductive assumption:

$$d((d\iota_p)\times\iota_q+(-1)^p\iota_p\times(d\iota_q)) = (d^2\iota_p)\times\iota_q+(-1)^{p-1}(d\iota_p)\times(d\iota_q)+(-1)^p(d\iota_p)\times(d\iota_q)+(-1)^q\iota_p\times(d^2\iota_q) = 0$$

because $d^2 = 0$. Note that this calculation would not have worked without the sign!

The subspace $\Delta^p \times \Delta^q \subseteq \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$ is convex and nonempty, and hence star-shaped. Therefore we know that $H_{p+q-1}(\Delta^p \times \Delta^q) = 0$ (remember, $p+q > 1$), which means that every cycle is a boundary. In other words, our necessary condition is also sufficient! So, choose any element with the right boundary and declare it to be $\iota_p \times \iota_q$.

The induction is now complete provided we can check that this choice satisfies naturality, bilinearity, and the Leibniz rule. I leave this as a relaxing exercise for the listener. □

The essential point here is that the space supporting the universal pair of simplices – $\Delta^p \times \Delta^q$ – has trivial homology. Naturality transports the result of that fact to the general situation.

The cross-product that this procedure constructs is not unique; it depends on a choice a choice of the chain $\iota_p \times \iota_q$ for each pair p, q with $p + q > 1$. The cone construction in the proof that star-shaped regions have vanishing homology provids us with a specific choice; but it turns out that any two choices are equivalent up to natural chain homotopy.

We return to homotopy invariance. To define our chain homotopy $h_X : S_n(X) \to S_{n+1}(X \times I)$, pick any 1-simplex $\iota : \Delta^1 \to I$ such that $d_0\iota = c_1^0$ and $d_1\iota = c_0^0$, and define

$$h_X\sigma = (-1)^n \sigma \times \iota \,.$$

Let's compute:

$$dh_X\sigma = (-1)^n d(\sigma \times \iota) = (-1)^n (d\sigma) \times \iota + \sigma \times (d\iota)$$

But $d\iota = c_1^0 - c_0^0 \in S_0(I)$, which means that we can continue (remembering that $|\partial\sigma| = n - 1$):

$$= -h_X d\sigma + (\sigma \times c_1^0 - \sigma \times c_0^0) = -h_X d\sigma + (\iota_{1*}\sigma - \iota_{0*}\sigma) \,,$$

using the normalization axiom of the cross-product. This is the result.

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