7 Homology cross product

In the last lecture we proved homotopy invariance of homology using the construction of a chain level bilinear cross-product

$$\times : S_p(X) \times S_q(Y) \to S_{p+q}(X \times Y)$$

that satisfied the Leibniz formula

$$d(a \times b) = (da) \times b + (-1)^p a \times (db)$$

What else does this map give us?

Let's abstract a little bit. Suppose we have three chain complexes A_*, B_*, and C_*, and suppose we have maps $\times : A_p \times B_q \to C_{p+q}$ that satisfy bilinearity and the Leibniz formula. What does this induce in homology?

Lemma 7.1. *These data determine a bilinear map* $\times : H_p(A) \times H_q(B) \to H_{p+q}(C)$.

Proof. Let $a \in Z_p(A)$ and $b \in Z_q(B)$. We want to define $[a] \times [b] \in H_{p+q}(C)$. We hope that $[a] \times [b] = [a \times b]$. We need to check that $a \times b$ is a cycle. By Leibniz, $d(a \times b) = da \times b + (-1)^p a \times db$, which vanishes becauxe a, b are cycles.

Now we need to check that homology class depends only on the homology classes we started with. So pick other cycles a' and b' in the same homology classes. We want $[a \times b] = [a' \times b']$. In other words, we need to show that $a \times b$ differs from $a' \times b'$ by a boundary. We can write $a' = a + d\bar{a}$ and $b' = b + d\bar{b}$, and compute, using bilinearity:

$$a' \times b' = (a + d\bar{a}) + (b + d\bar{b}) = a \times b + a \times d\bar{b} + (d\bar{a}) \times b + (d\bar{a}) \times (d\bar{b})$$

We need to deal with the last three terms here. But since $da = 0$,

$$d(a \times \bar{b}) = (-1)^p a \times (d\bar{b})\,.$$

Since $d\bar{b} = 0$,

$$d((\bar{a}) \times b) = (d\bar{a}) \times b\,.$$

And since $d^2\bar{b} = 0$,

$$d(a \times \bar{b}) = (d\bar{a}) \times (d\bar{b})\,.$$

This means that $a' \times b'$ and $a \times b$ differ by

$$d\left((-1)^p(a \times \bar{b}) + \bar{a} \times b + \bar{a} \times d\bar{b}\right)\,,$$

and so are homologous.

The last step is to check bilinearity, which is left to the listener. □

This gives the following result.

Theorem 7.2. *There is a map*

$$\times : H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y)$$

that is natural, bilinear, and normalized.

We will see that this map is also *uniquely defined* by these conditions, unlike the chain-level cross product.

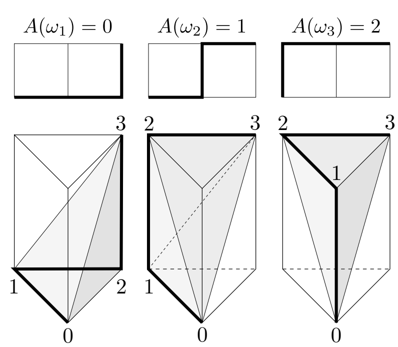
I just want to mention an explicit choice of $\iota_p \times \iota_q$. This is called the Eilenberg-Zilber chain. You're highly encouraged to think about this yourself. It comes from a triangulation of the prism.

The simplices in this triangulation are indexed by order preserving injections

$$\omega : [p+q] \to [p] \times [q]$$

Injectivity forces $\omega(0) = (0,0)$ and $\omega(p+q) = (p,q)$. Each such map determines an affine map $\Delta^{p+q} \to \Delta^p \times \Delta^q$ of the same name. These will be the singular simplices making up $\iota_p \times \iota_q$. To specify the coefficients, think of ω as a staircase in the rectangle $[0,p] \times [0,q]$. Let $A(\omega)$ denote the area under that staircase. Then the Eilenberg-Zilber chain is given by

$$\iota_p \times \iota_q = \sum (-1)^{A(\omega)} \omega$$



This chain is due to Eilenberg and Mac Lane; the description appears in a paper [4] by Eilenberg and Moore. It's very pretty, but it's combinatorially annoying to check that this satisfies the conditions of the theorem. It provides an explicit chain map

$$\beta_{X,Y} : S_*(X) \times S_*(Y) \to S_*(X \times Y)$$

that satisfies many good properties on the nose and not just up to chain homotopy. For example, it's *associative* –

$$\begin{array}{ccc} S_*(X) \times S_*(Y) \times S_*(Z) & \xrightarrow{\beta_{X,Y} \times 1} & S_*(X \times Y) \times S_*(Z) \\ \Big\downarrow {\scriptstyle 1 \times \beta Y,Z} & & \Big\downarrow {\scriptstyle \beta_{X \times Y, Z}} \\ S_*(X) \times S_*(Y \times Z) & \xrightarrow{\beta_{X, Y \times Z}} & S_*(X \times Y \times Z) \end{array}$$

commutes – and *commutative* –

$$\begin{array}{ccc} S_*(X) \times S_*(Y) & \xrightarrow{\beta_{X,Y}} & S_*(X \times Y) \\ \Big\downarrow {\scriptstyle T} & & \Big\downarrow {\scriptstyle S_*(T)} \\ S_*(Y) \times S_*(X) & \xrightarrow{\beta_{Y,X}} & S_*(X \times Y) \end{array}$$

commutes, where on spaces $T(x, y) = (y, x)$, and on chain complexes $T(a, b) = (-1)^{pq}(b, a)$ when a has degree p and b has degree q.

We will see that these properties hold up to chain homotopy for any choice of chain-level cross product.

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