

## Chapter 3

# Vector bundles and principal bundles

### 16 Vector bundles

Each point in a smooth manifold  $M$  has a “tangent space.” This is a real vector space, whose elements are equivalence classes of smooth paths  $\sigma : \mathbb{R} \rightarrow M$  such that  $\sigma(0) = x$ . The equivalence relation retains only the velocity vector at  $t = 0$ . These vector spaces “vary smoothly” over the manifold. The notion of a vector bundle is a topological extrapolation of this idea.

Let  $B$  be a topological space. To begin with, let’s define the “category of spaces over  $B$ ,”  $\mathbf{Top}/B$ . An object is just a map  $E \rightarrow B$ . To emphasize that this is single object, and that it is an object “over  $B$ ,” we may give it a symbol and display the arrow vertically:  $\xi : E \downarrow B$ . A morphism from  $p' : E' \rightarrow B$  to  $p : E \rightarrow B$  is a map  $E' \rightarrow E$  making

$$\begin{array}{ccc} E' & \xrightarrow{\quad} & E \\ & \searrow p' & \swarrow p \\ & & B \end{array}$$

commute.

This category has products, given by the fiber product over  $B$ :

$$E' \times_B E = \{(e', e) : p'e' = pe\} \subseteq E' \times E.$$

Using it we can define an “abelian group over  $B$ ”: an object  $E \downarrow B$  together with a “zero section”  $0 : B \rightarrow E$  (that is, a map from the terminal object of  $\mathbf{Top}/B$ ) and an “addition”  $E \times_B E \rightarrow E$  (of spaces over  $B$ ) satisfying the usual properties.

As an example, any topological abelian group  $A$  determines an abelian group over  $B$ , namely  $\text{pr}_1 : B \times A \rightarrow B$  with its evident structure maps. If  $A$  is a ring, then  $\text{pr}_1 : B \times A \rightarrow B$  is a “ring over  $B$ .” For example, we have the “reals over  $B$ ,” and hence can define a “vector space over  $B$ .” Each fiber has the structure of a vector space, and this structure varies continuously as you move around in the base.

Vector spaces over  $B$  form a category in which the morphisms are maps covering the identity map of  $B$  that are linear on each fiber.

**Example 16.1.** Let  $S$  be the subspace of  $\mathbb{R}^2$  consisting of the  $x$  and  $y$  axes, and consider  $\text{pr}_1 : S \rightarrow \mathbb{R}$ . Then  $\text{pr}_1^{-1}(0) = \mathbb{R}$  and  $\text{pr}_1^{-1}(s) = 0$  for  $s \neq 0$ . With the evident structure maps, this is a perfectly good (“skyscraper”) vector space over  $\mathbb{R}$ . This example is peculiar, however; it is not locally constant. Our definition of vector bundles will exclude it and similar oddities. Sheaf theory is the proper home for examples like this.

But this example occurs naturally even if you restrict to trivial bundles and maps between them. The trivial bundle  $\text{pr}_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  has as an endomorphism the map

$$(s, t) \mapsto (s, st).$$

This map is an isomorphism on almost all fibers, but is zero over  $s = 0$ . So if you want to form a kernel or the cokernel, you will get the skyscraper vector space over  $\mathbb{R}$ . The image will be a vector space over  $X$  with a complementary peculiarity.

**Definition 16.2.** A *vector bundle* over  $B$  is a vector space  $E$  over  $B$  that is locally trivial – that is, every point  $b \in B$  has a neighborhood over which  $E$  is isomorphic to a trivial bundle – and whose fiber vector spaces are all of finite dimension.

**Remark 16.3.** As in our definition of fiber bundles, we will always assume that a vector bundle admits a numerable trivializing cover. On the other hand, there is nothing to stop us from replacing  $\mathbb{R}$  with  $\mathbb{C}$  or even with the quaternions  $\mathbb{H}$ , and talking about complex or quaternionic vector bundles.

If  $\xi : E \downarrow B$  is a vector bundle, then  $E$  is called the *total space*, the map  $p : E \rightarrow B$  is called the *projection map*, and  $B$  is called the *base space*. We may write  $E(\xi), B(\xi)$  for the total space and base space, and  $\xi_b$  for the fiber of  $\xi$  over  $b \in B$ .

If all the fibers are of dimension  $n$ , we have an  $n$ -dimensional vector bundle or an “ $n$ -plane bundle.”

**Example 16.4.** The “trivial”  $n$ -dimensional vector bundle over  $B$  is the projection  $\text{pr}_1 : B \times \mathbb{R}^n \rightarrow B$ . We may write  $n\epsilon$  for it.

**Example 16.5.** At the other extreme, Grassmannians support highly nontrivial vector bundles. We can form Grassmannians over any one of the three (skew)fields  $\mathbb{R}, \mathbb{C}, \mathbb{H}$ . Write  $K$  for one of them, and consider the (left)  $K$ -vector space  $K^n$ . The *Grassmannian* (or *Grassmann manifold*)  $\text{Gr}_k(K^n)$  is the space of  $k$ -dimensional  $K$ -subspaces of  $K^n$ . As we saw last term, this is a topologized as a quotient space of a Stiefel variety  $V_k(K^n)$  of  $k$ -frames in  $K^n$ . To each point in  $\text{Gr}_k(K^n)$  is associated a  $k$ -dimensional subspace of  $K^n$ . This provides us with a  $k$ -dimensional  $K$ -vector bundle  $\xi_{n,k}$  over  $\text{Gr}_k(K^n)$ , with total space

$$E(\xi_{n,k}) = \{(V, x) \in \text{Gr}_k(K^n) \times K^n : x \in V\}$$

This is the *canonical* or *tautologous* vector bundle over  $\text{Gr}_k(K^n)$ . It occurs as a subbundle of  $n\epsilon$ .

**Exercise 16.6.** Prove that  $\xi_{n,k}$ , as defined above, is locally trivial, so is a vector bundle over  $\text{Gr}_k(K^n)$ .

For instance, when  $k = 1$ , we have  $\text{Gr}_1(\mathbb{R}^n) = \mathbf{R}P^{n-1}$ . The tautologous bundle  $\xi_{n,1}$  is 1-dimensional; it is a *line bundle*, the canonical line bundle over  $\mathbf{R}P^{n-1}$ . We may write  $\lambda$  for this or any line bundle.

**Example 16.7.** Let  $M$  be a smooth manifold. Define  $\tau_M$  to be the tangent bundle  $TM \downarrow M$  over  $M$ . For example, if  $M = S^{n-1}$ , then

$$TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbb{R}^n : v \cdot x = 0\}.$$

### Constructions with vector bundles

Just about anything that can be done for vector spaces can also be done for vector bundles:

1. The pullback of a vector bundle is again a vector bundle: If  $p : E \rightarrow B$  is a vector bundle then the map  $p'$  in the pullback diagram below is also a vector bundle.

$$\begin{array}{ccc} E' & \xrightarrow{\bar{f}} & E \\ \downarrow p' & & \downarrow p \\ B' & \xrightarrow{f} & B \end{array}$$

The pullback of  $\xi : E \downarrow B$  bundle may be denoted  $f^*\xi$ .

There's a convenient way characterize a pullback: the top map  $\bar{f}$  in the pullback diagram has two key properties: It covers  $f$ , and it is a linear isomorphism on fibers. These conditions suffice to present  $p'$  as the pullback of  $p$  along  $f$ .

2. If  $p : E \rightarrow B$  and  $p' : E' \rightarrow B'$ , then the product map  $p \times p' : E \times E' \rightarrow B \times B'$  is a vector bundle whose fiber over  $(x, y)$  is the vector space  $p^{-1}(x) \times p'^{-1}(y)$ .
3. If  $B = B'$ , we can form the pullback:

$$\begin{array}{ccc} E \oplus E' & \longrightarrow & E \times E' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\Delta} & B \times B \end{array}$$

The bundle  $\xi \oplus \xi' : E \oplus E' \downarrow B$  is called the *Whitney sum* of  $\xi : E \downarrow B$  and  $\xi' : E' \downarrow B$ . (Hassler Whitney (1907–1989) working mainly at the Institute for Advanced Study in Princeton, is responsible for many early ideas in geometric topology.) For instance,

$$n\epsilon = \epsilon \oplus \cdots \oplus \epsilon.$$

4. If  $\xi : E \downarrow B$  and  $\xi' : E' \downarrow B$  are two vector bundles over  $B$ , we can form another vector bundle  $\xi \otimes \xi'$  over  $B$  by taking the fiberwise tensor product. Likewise, taking the fiberwise Hom produces a vector bundle  $\text{Hom}(\xi, \xi')$  over  $B$ .

**Example 16.8.** Recall from Example 16.5 the tautological bundle  $\lambda$  over  $\mathbf{R}P^{n-1}$ . The tangent bundle  $\tau_{\mathbf{R}P^{n-1}}$  also lives over  $\mathbf{R}P^{n-1}$ . It is natural to wonder what is the relationship between these two bundles. We claim that

$$\tau_{\mathbf{R}P^{n-1}} = \text{Hom}(\lambda, \lambda^\perp)$$

where  $\lambda^\perp$  denotes the fiberwise orthogonal complement of  $\lambda$  in  $n\epsilon$ . To see this, make use of the double cover  $S^{n-1} \downarrow \mathbf{R}P^{n-1}$ . The projection map is smooth, and covered by a fiberwise isomorphism of tangent bundles. The fibers  $T_x S^{n-1}$  and  $T_{-x} S^{n-1}$  are both identified with the orthogonal complement of  $\mathbb{R}x$  in  $\mathbb{R}^n$ , and the differential of the antipodal map sends  $v$  to  $-v$ . So the tangent vector to  $\pm x \in \mathbf{R}P^{n-1}$  represented by  $(x, v)$  is the same as the tangent vector represented by  $(-x, -v)$ . This tangent vector determines a homomorphism  $\lambda_x \rightarrow \lambda_x^\perp$  sending  $tx$  to  $tv$ .

**Exercise 16.9.** Prove that

$$\tau_{\text{Gr}_k(\mathbb{R}^n)} = \text{Hom}(\xi_{n,k}, \xi_{n,k}^\perp).$$

## Metrics and splitting exact sequences

A map of vector bundles,  $\xi \rightarrow \eta$ , over a fixed base can be identified with a section of  $\text{Hom}(\xi, \eta)$ . We have seen that the kernel and cokernel of a homomorphism will be vector bundles only if the rank is locally constant.

In particular, we can form kernels of surjections and cokernels of injections; and consider short exact sequences of vector bundles. It is a characteristic of topology, as opposed to analytic or algebraic geometry, that short exact sequences of vector bundles always split. To see this we use a “metric.”

**Definition 16.10.** A *metric* on a vector bundle is a continuous choice of inner products on the fibers.

**Lemma 16.11.** Any (numerable) vector bundle  $\xi$  admits a metric.

*Proof.* This will use the fact that if  $g, g'$  are both inner products on a vector space then  $tg + (1-t)g'$  (for  $t$  between 0 and 1) is another. So the space of metrics on a vector bundle  $E \downarrow B$  forms a convex subset of the vector space of continuous functions  $E \times_B E \rightarrow \mathbb{R}$ .

Pick a trivializing open cover  $\mathcal{U}$  for  $\xi$ , and for each  $U \in \mathcal{U}$  an isomorphism  $\xi|_U \cong U \times V_U$ . Pick an inner product  $g_U$  on each of the vector spaces  $V_U$ . Pick a partition of unity subordinate to  $\mathcal{U}$ ; that is, functions  $\phi_U : U \rightarrow [0, 1]$  such that the preimage of  $(0, 1]$  is  $U$  and

$$\sum_{x \in U} \phi_U(x) = 1.$$

Now the sum

$$g = \sum_U \phi_U g_U$$

is a metric on  $\xi$ . □

**Corollary 16.12.** Any exact sequence  $0 \rightarrow \xi' \rightarrow \xi \rightarrow \xi'' \rightarrow 0$  of vector bundles (over the same base) splits.

*Proof.* Pick a metric for  $\xi$ . Using it, form the orthogonal complement  $\xi'^{\perp}$ . The composite

$$\xi'^{\perp} \hookrightarrow \xi \rightarrow \xi''$$

is an isomorphism. This provides a splitting of the surjection  $\xi \rightarrow \xi''$  and hence of the short exact sequence. □

## 17 Principal bundles, associated bundles

### *I*-invariance

We will denote by  $\text{Vect}(B)$  the set of isomorphism classes of vector bundles over  $B$ , and  $\text{Vect}_n(B)$  the set of  $n$ -plane bundles.

**Exercise 17.1.** Justify the use of the word “set”!

Vector bundles pull back, and isomorphic vector bundles pull back to isomorphic vector bundles. This establishes  $\mathbf{Vect}$  as a contravariant functor on  $\mathbf{Top}$ :

$$\mathbf{Vect} : \mathbf{Top}^{op} \rightarrow \mathbf{Set} .$$

How computable is this functor? As a first step in answering this, we note that it satisfies the following characteristic property of bundle theories.

**Theorem 17.2.** *The functor  $\mathbf{Vect}$  is  $I$ -invariant (where  $I$  denotes the unit interval): that is, the projection  $\text{pr}_1 : X \times I \rightarrow X$  induces an isomorphism  $\mathbf{Vect}(X) \rightarrow \mathbf{Vect}(X \times I)$ .*

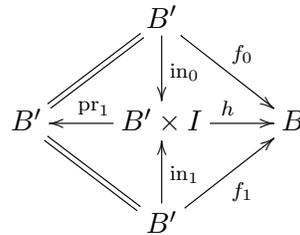
We will prove this in the next lecture. The map  $\text{pr}_1 : X \times I \rightarrow X$  is a split surjection, so  $\text{pr}_1^* : \mathbf{Vect}(X) \rightarrow \mathbf{Vect}(X \times I)$  is a split injection. Surjectivity is harder.

An important corollary of this result is:

**Corollary 17.3.**  *$\mathbf{Vect}$  is a homotopy functor.*

*Proof.* Let  $\xi : E \downarrow B$  be a vector bundle and suppose  $H : B' \times I \rightarrow B$  a homotopy between two maps  $f_0$  and  $f_1$ . We are claiming that  $f_0^* \xi \cong f_1^* \xi$ . This is far from obvious!

In the diagram



the map  $\text{pr}_1$  induces a surjection in  $\mathbf{Vect}$  by Theorem 17.2. It follows that  $\text{in}_0^* = \text{in}_1^*$ , so  $f_0^* = \text{in}_0^* \circ h^* = \text{in}_1^* \circ h^* = f_1^*$ . □

## Principal bundles

**Definition 17.4.** Let  $G$  be a topological group. A *principal  $G$ -bundle* is a right action of  $G$  on a space  $P$  such that:

1.  $G$  acts freely.
2. The orbit projection  $P \rightarrow P/G$  is a fiber bundle.

There's a famous video of J.-P. Serre talking about writing mathematics. In it he says you have to know the difference between “principle” and “principal”. He contemplated what a “bundle of principles” might be – varying over a moduli space of individuals, perhaps.

We will only care about Lie groups, among which are discrete groups.

Principal bundles are not unfamiliar objects, as the next example shows.

**Example 17.5.** Suppose  $G$  is discrete. Then the fibers of the orbit projection  $P \rightarrow P/G$  are all discrete. Therefore, the condition that  $P \rightarrow P/G$  is a fiber bundle is simply that it's a covering projection. Such an action is sometimes said to be “properly discontinuous.”

As a special case, let  $X$  be a space with universal cover  $\tilde{X} \downarrow X$  (so  $X$  is path connected and semi-locally simply connected). Then  $\pi_1(X)$  acts freely on  $\tilde{X}$ , and  $p : \tilde{X} \rightarrow X$  is the orbit projection; we have a principal  $\pi_1(X)$ -bundle. Explicit examples include the principal  $C_2$ -bundles  $S^{n-1} \downarrow \mathbf{R}P^{n-1}$ .

We can use the universal cover to classify covering spaces of  $X$ . Remember how this goes: The fundamental group at  $*$  acts on the fiber over  $*$  of any covering projection to produce a left  $\pi_1(X)$ -set. A functor in the other direction is given as follows. Let  $F$  be any set with left  $\pi_1(X)$ -action, and form the “balanced product”

$$\tilde{X} \times_{\pi_1(X)} F = \tilde{X} \times F / \sim$$

where  $(y, gz) \sim (yg, z)$ , for elements  $y \in \tilde{X}$ ,  $z \in F$ , and  $g \in \pi_1(X)$ . The composite  $p \circ \text{pr}_1 : \tilde{X} \times F \rightarrow X$  factors to give a map

$$\tilde{X} \times_{\pi_1(X)} F \rightarrow X$$

that is a covering projection.

**Theorem 17.6** (Covering space theory). *Suppose that  $X$  is path-connected and semi-locally simply connected. Then these constructions provide an equivalence of categories*

$$\left\{ \begin{array}{c} \text{Left } \pi_1(X)\text{-sets} \\ \text{equivariant bijections} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Covering spaces of } X \\ \text{isomorphisms} \end{array} \right\}.$$

This story motivates constructions in the more general setting of principal  $G$ -bundles.

**Construction 17.7.** Let  $P \downarrow B$  be a principal  $G$ -bundle. If  $F$  is a left  $G$ -space, we can define a new fiber bundle, “associated” to  $P \downarrow B$ , exactly as above:

$$\begin{array}{c} P \times_G F \\ \downarrow q \\ B \end{array}$$

Let’s check that the fibers are homeomorphic to  $F$ . Let  $x \in B$ , and pick  $y \in P$  over  $x$ . Map  $F \rightarrow q^{-1}(x)$  by  $z \mapsto [y, z]$ . We claim that this is a homeomorphism. The inverse  $q^{-1}(x) \rightarrow F$  is given by

$$[y', z'] = [y, gz'] \mapsto gz',$$

where  $y' = yg$  for some  $g$  (which is necessarily unique since the  $G$  action is simply transitive on fibers of  $P$ ). These two maps are inverse homeomorphisms.

If  $F$  is a finite dimensional vector space on which  $G$  acts linearly, then we get a vector bundle from this construction.

Let  $\xi : E \downarrow B$  be an  $n$ -plane bundle. Construct a principal  $GL_n(\mathbb{R})$ -bundle  $P(\xi)$  by defining

$$P(\xi)_b = \{\text{ordered bases for } E(\xi)_b = \text{Iso}(\mathbb{R}^n, E(\xi)_b)\}.$$

To define the topology, think of  $P(\xi)$  as a quotient of the disjoint union of trivial bundles over the open sets in a trivializing cover for  $\xi$ ; while for trivial bundles

$$P(B \times \mathbb{R}^n) = B \times \text{Iso}(\mathbb{R}^n, \mathbb{R}^n)$$

topologically, where  $\text{Iso}(\mathbb{R}^n, \mathbb{R}^n) = GL_n(\mathbb{R})$  is given the usual topology as a subspace of  $\mathbb{R}^{n^2}$ .

There is a right action of  $GL_n(\mathbb{R})$  on  $P(\xi)$ , given by precomposition. It is easy to see that this action is free and simply transitive on fibers. One therefore has a principal action of  $GL_n(\mathbb{R})$  on  $P(\xi)$ . The bundle  $P(\xi)$  is called the *principalization* of  $\xi$ .

Given the principalization  $P(\xi)$ , we can recover the total space  $E(\xi)$ , using the defining linear action of  $GL_n(\mathbb{R})$  on  $\mathbb{R}^n$ :

$$E(\xi) \cong P(\xi) \times_{GL_n(\mathbb{R})} \mathbb{R}^n.$$

These two constructions are inverses: the theories of  $n$ -plane bundles and of principal  $GL_n(\mathbb{R})$ -bundles are equivalent.

**Remark 17.8.** Suppose that we have a metric on  $\xi$ . Instead of looking at all ordered bases, we can use instead all ordered orthonormal bases in each fiber. This give the *frame bundle*

$$\text{Fr}(\xi)_b = \{\text{ordered orthonormal bases of } E(\xi)_b\} = \{\text{isometric isomorphisms } \mathbb{R}^n \rightarrow E(\xi)_b\}.$$

The orthogonal group  $O(n)$  acts freely and fiberwise transitively on this space, endowing  $\text{Fr}(\xi)$  with the structure of a principal  $O(n)$ -bundle.

Providing a vector bundle with a metric, when viewed in terms of the associated principal bundles, is an example of “reduction of the structure group.” We are giving a principal  $O(n)$  bundle  $P$  together with an isomorphism of principal  $GL_n(\mathbb{R})$  bundles from  $P \times_{O(n)} GL_n(\mathbb{R})$  to the principalization of  $\xi$ . Many other geometric structures can be described in this way. An orientation of  $\xi$ , for example, consists of a principal  $SL_n(\mathbb{R})$  bundle  $Q$  together with an isomorphism from  $Q \times_{SL_n(\mathbb{R})} GL_n(\mathbb{R})$  to the principalization of  $\xi$ .

Fix a topological group  $G$ . Define  $\text{Bun}_G(B)$  as the set of isomorphism classes of  $G$ -bundles over  $B$ . An isomorphism is a  $G$ -equivariant homeomorphism over the base. Again, arguing as above, this leads to a contravariant functor  $\text{Bun}_G : \mathbf{Top} \rightarrow \mathbf{Set}$ . The above discussion gives a natural isomorphism of functors:

$$\text{Bun}_{GL_n(\mathbb{R})}(B) \cong \text{Vect}(B).$$

The *I*-invariance of  $\text{Vect}$  is therefore a special case of:

**Theorem 17.9.**  $\text{Bun}_G$  is *I*-invariant, and hence is a homotopy functor.

One case is easy to prove: If  $X$  is contractible, then any principal  $G$ -bundle  $P \downarrow X$  is trivial. It’s enough to construct a section. Since the identity map on  $X$  is homotopic to a constant map (with value  $*$   $\in X$ , say), the constant map  $c_p : X \rightarrow P$  for any  $p \in P$  over  $*$   $\in X$  makes

$$\begin{array}{ccc} & & P \\ & \nearrow c_p & \downarrow \\ X & \longrightarrow & X \end{array}$$

commute up to homotopy. But since  $P \downarrow X$  is a fibration, this implies that there is then an *actual* section. And a section of a principal bundle determines a trivialization of it.

We have considered only isomorphisms of principal bundles. But any continuous equivariant map of principal bundles over the same base that covers the identity endomorphism of the base is in fact an isomorphism.

## 18 *I*-invariance of $\text{Bun}_G$ , and *G*-CW-complexes

Let  $G$  be a topological group. We want to show that the functor  $\text{Bun}_G : \mathbf{Top}^{op} \rightarrow \mathbf{Set}$  is *I*-invariant, i.e., the projection  $\text{pr}_1 : X \times I \rightarrow X$  induces an isomorphism  $\text{Bun}_G(X) \xrightarrow{\cong} \text{Bun}_G(X \times I)$ .

Injectivity is easy: the composite  $X \xrightarrow{\text{in}_0} X \times I \xrightarrow{\text{pr}_1} X$  is the identity and gives you a splitting  $\text{Bun}_G(X) \xrightarrow{\text{pr}_1^*} \text{Bun}_G(X \times I) \xrightarrow{\text{in}_0^*} \text{Bun}_G(X)$ .

The rest of this lecture is devoted to proving surjectivity. There are various ways to do this. Husemoller does the general case; see [13, §4.9]. Steve Mitchell has a nice treatment in [28]. We will prove this when  $X$  is a CW-complex, by adapting CW methods to the equivariant situation.

To see the point of this approach, notice that the word “free” is used somewhat differently in the context of group actions than elsewhere. The left adjoint of the forgetful functor from  $G$ -spaces to spaces sends a space  $X$  to the  $G$ -space  $X \times G$  in which  $G$  acts, from the right, by  $(x, g)h = (x, gh)$ . If  $G$  and  $X$  are discrete, any free action of  $G$  on  $X$  has this form. But this is not true topologically: just think of the antipodal action of  $C_2$  on the circle, for instance.

The condition that an action is principal is one way to demand that an action should be “locally” free in the stronger sense.  $G$ -CW complexes afford a different way.

### $G$ -CW-complexes

We would like to set up a theory of CW-complexes with an action of the group  $G$ . The relevant question is, “What is a  $G$ -cell?” There is a choice here. For us, and for the standard definition of a  $G$ -CW-complex, the right thing to say is that it is a  $G$ -space of the form

$$D^n \times H \backslash G.$$

Here  $H$  is a closed subgroup of  $G$ , and  $H \backslash G$  is the orbit space of the action of  $H$  on  $G$  by left translation, viewed as a right  $G$ -space. The “boundary” of the  $G$ -cell  $D^n \times H \backslash G$  is just  $\partial D^n \times H \backslash G$  (with the usual convention that  $\partial D^0 = \emptyset$ ).

**Definition 18.1.** A *relative  $G$ -CW-complex* is a (right)  $G$ -space  $X$  with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X$$

by  $G$ -subspaces such that for all  $n \geq 0$  there exists a pushout square of  $G$ -spaces

$$\begin{array}{ccc} \coprod \partial D_i^n \times H_i \backslash G & \longrightarrow & \coprod D_i^n \times H_i \backslash G \\ \downarrow & & \downarrow \\ X_{n-1} & \longrightarrow & X_n, \end{array}$$

and  $X$  has the direct limit topology.

**Remarks 18.2.** A CW-complex is just a  $G$ -CW-complex for the trivial group  $G$ . If  $G$  is discrete, the skeleton filtration provides  $X$  with the structure of a CW-complex by neglect of the  $G$ -action. The subspace  $X_n$  is called the  $n$ -skeleton of  $X$ , even though if  $G$  is itself of positive dimension  $X_n$  may well have dimension larger than  $n$ .

If  $X$  is a  $G$ -CW-complex, then  $X/G$  inherits a CW-structure whose  $n$ -skeleton is given by  $(X/G)_n = X_n/G$ .

If  $P \downarrow X$  is a principal  $G$ -bundle, a CW-structure on  $X$  lifts to a  $G$ -CW-structure on  $P$ .

The action of  $G$  on a  $G$ -CW complex is principal if and only if all the isotropy groups are trivial.

A good source for much of this is [18]; see for example Remark 2.8 there.

**Theorem 18.3** (Illman [14], Verona). *If  $G$  is a compact Lie group and  $M$  a smooth manifold on which  $G$  acts by diffeomorphisms, then  $M$  admits a  $G$ -CW structure.*

It's quite challenging in general to write down a *G*-CW structure even in simple cases, such as when the manifold is the unit sphere in an orthogonal representation of *G*. But sometimes it's easy. For example, the standard CW structure on  $\mathbb{R}P^{n-1}$ , with one *k*-cell for each *k* with  $0 \leq k \leq n-1$ , lifts to a  $C_2$ -CW structure on  $S^{n-1}$ . In it, the (*k* - 1)-skeleton is  $S^{k-1}$ , for each  $k \leq n$ , and there are two *k*-cells, given by the upper and lower hemisphere of  $S^k$ .

For another example, regard  $S^1$  as the complex numbers of magnitude 1, equipped with a  $C_2$  action by complex conjugation. This has a  $C_2$ -CW structure with 0-skeleton given by  $\{\pm 1\}$  a single free 1-cell.

### Proof of *I*-invariance

Recall that our goal is to prove that every principal *G*-bundle  $p : P \rightarrow X \times I$  is pulled back from some principle *G*-bundle over *X*. Actually there's no choice here; since  $pr_1 \circ \text{in}_0 = 1$ , *P* must be pulled back from  $\text{in}_0^*P$ , that is, from the restriction of *P* to  $X \times 0$ .

For notational convenience, let us write  $Y = X \times I$ . Remember that we are assuming that *X* is a CW-complex. We will filter *Y* by subcomplexes, as follows. Let  $Y_0 = X \times 0$ ; in general, we define

$$Y_n = X_n \times 0 \cup X_{n-1} \times I.$$

We may construct  $Y_n$  from  $Y_{n-1}$  via a pushout:

$$\begin{array}{ccc} \coprod(\partial D^{n-1} \times I \cup D^{n-1} \times 0) & \longrightarrow & \coprod(D^{n-1} \times I) \\ \downarrow & & \downarrow \\ Y_{n-1} & \longrightarrow & Y_n, \end{array}$$

The restriction of *P* to  $Y_n$  is a principal bundle with total space

$$P_n = p^{-1}(Y_n).$$

So  $P_0 \downarrow Y_0$  is just  $\text{in}_0^*P \downarrow X$ .

We will show that *P* and  $\text{pr}_1^*\text{in}_0^*P$  are isomorphic over *Y*. For this it will be enough to construct an equivariant map  $P \rightarrow \text{in}_0^*P$  covering the projection map  $\text{pr}_1 : Y \rightarrow X$ . We'll do this by inductively constructing compatible equivariant maps  $P_n \rightarrow P_0$  covering the composites  $Y_n \hookrightarrow Y \rightarrow X$ , starting with the identity map  $P_0 \rightarrow \text{in}_0^*P$  covering the isomorphism  $Y_0 \rightarrow X$ .

We can build  $P_n$  from  $P_{n-1}$  by lifting the pushout construction of  $Y_n$  from  $Y_{n-1}$ :

$$\begin{array}{ccc} \coprod(\partial D^{n-1} \times I \cup D^{n-1} \times 0) \times G & \longrightarrow & \coprod(D^{n-1} \times I) \times G \\ \downarrow & & \downarrow \\ P_{n-1} & \longrightarrow & P_n \end{array}$$

So to extend  $P_{n-1} \rightarrow P_0$  to  $P_n \rightarrow P_0$ , we must construct an equivariant map  $f$  in

$$\begin{array}{ccc}
 \coprod(\partial D^{n-1} \times I \cup D^{n-1} \times 0) & \longrightarrow & \coprod(D^{n-1} \times I) \\
 \downarrow & & \downarrow \\
 \coprod(\partial D^{n-1} \times I \cup D^{n-1} \times 0) \times G & \longrightarrow & \coprod(D^{n-1} \times I) \times G \\
 \downarrow & & \downarrow \\
 P_{n-1} & \longrightarrow & P_n \\
 & \searrow & \downarrow \\
 & & P_0
 \end{array}
 \quad (3.1)$$

$\swarrow$  (dotted arrow)  $f$   $\searrow$  (dotted arrow)

covering the map  $Y_n \rightarrow Y_0$ . Since the action is free, it's enough to define  $f$  on  $D^{n-1} \times I$  for each cell, in such a way that the diagram

$$\begin{array}{ccc}
 \partial D^{n-1} \times I \cup D^{n-1} \times 0 & \longrightarrow & D^{n-1} \times I \\
 \downarrow & & \downarrow \\
 P_{n-1} & \longrightarrow & P_0 \\
 \downarrow & & \downarrow \\
 Y_{n-1} & \longrightarrow & Y_0
 \end{array}$$

$\swarrow$  (dotted arrow)  $\searrow$  (dotted arrow)

commutes, and then extend by equivariance. Since

$$(D^{n-1} \times I, \partial D^{n-1} \times I \cup D^{n-1} \times 0) \cong (D^{n-1} \times I, D^{n-1} \times 0),$$

what we have is:

$$\begin{array}{ccc}
 D^{n-1} \times 0 & \longrightarrow & P_0 \\
 \downarrow & \nearrow & \downarrow \\
 D^{n-1} \times I & \longrightarrow & Y_0
 \end{array}$$

So the dotted map exists, since  $P_0 \rightarrow Y_0$  is a fibration! □

## 19 The classifying space of a group

### Representability

**Theorem 19.1.** *Let  $G$  be a topological group and  $\xi : E \downarrow B$  a principal  $G$ -bundle such that  $E$  is weakly contractible (just as a space, forgetting the  $G$ -action). For any CW complex  $X$ , the map*

$$[X, B] \rightarrow \text{Bun}_G(X)$$

*sending a map  $f : X \rightarrow B$  to the isomorphism class of  $f^*\xi$  is bijective.*

This theorem has two parts: surjectivity and injectivity. Both are proved using the following proposition.

**Proposition 19.2.** *Let  $E$  be a  $G$ -space that is weakly contractible as a space. Let  $(P, A)$  be a free relative  $G$ -CW complex. Then any equivariant map  $f : A \rightarrow E$  extends to an equivariant map  $P \rightarrow E$ , and this extension is unique up to an equivariant homotopy rel  $A$ .*

*Proof.* Just do what comes naturally, after the experience of the proof of  $I$ -invariance! □

*Proof of Theorem 19.1.* Surjectivity is immediate; take  $A = \emptyset$ .

To prove injectivity, let  $f_0, f_1 : P \rightarrow E$  be two equivariant maps. We wish to show that they are homotopic by an equivariant homotopy, which thus descends to a homotopy between the induced maps on orbit spaces. Our data give an equivariant map  $A = P \times \partial I \rightarrow E$ , which we extend to an equivariant map from  $P \times I$  again using Proposition 19.2. □

As usual, the representing object is unique up to isomorphism (in the homotopy category). Any choice of contractible free  $G$ -CW complex will be written  $EG$ , and its orbit space  $BG$ .  $EG \downarrow BG$  is the universal principal  $G$ -space, and  $BG$  classifies principal  $G$ -bundles.

What remains is to *construct* a  $G$ -CW complex that is both free and contractible. There are many ways to do this. One can use Brown Representability, for example.

When the group is discrete, say  $\pi$ , this amounts to finding a  $K(\pi, 1)$ : the action of  $\pi$  on the universal cover is “properly discontinuous,” which is to say principal. So we have a bunch of examples! For instance, let  $\pi = \pi_1(\Sigma)$  where  $\Sigma$  is any closed connected surface other than  $S^2$  and  $\mathbb{R}P^2$ . Then any principal  $\pi$ -bundle over any CW-complex  $B$  is pulled back from the universal cover of  $\Sigma$  under a unique homotopy class of maps  $B \rightarrow \Sigma$ .

If  $G$  is a compact Lie group – for example a finite group – there is a very geometric way to go about this, based on the following result.

**Theorem 19.3** (Peter-Weyl, [17, Corollary IV.4.22]). *Any compact Lie group admits a finite-dimensional faithful unitary representation.*

Clearly, if  $P$  is free as a  $G$ -space then it is also free as an  $H$ -space for any subgroup  $H$  of  $G$ . It’s also the case that if  $P$  is a principal  $G$ -space then it is also a principal  $H$  space, provided that  $H$  is a *closed* subgroup of  $G$ .

Combining these facts, we see that in order to construct a universal principal  $G$  action, for any compact Lie group  $G$ , it suffices to construct such a thing for the particular Lie groups  $U(n)$ .

## Gauss maps

Before we look for highly connected spaces on which  $U(n)$  acts, let’s look at the case in which the base space is a compact Hausdorff space (for example a finite complex). In this case we can be more geometrically explicit about the classifying map.

**Lemma 19.4.** *Over a compact Hausdorff space, any vector bundle embeds in a trivial bundle.*

*Proof.* Let  $\mathcal{U}$  be a trivializing open cover of the base  $B$ ; since  $B$  is compact, we may assume that  $\mathcal{U}$  is finite, with, say,  $k$  elements  $U_1, \dots, U_k$ . We agreed that our vector bundles would always be numerable, but we don’t even have to mention this here since compact Hausdorff spaces are paracompact. So we can choose a partition of unity  $\{\phi_i\}$  subordinate to  $\mathcal{U}$ . By treating path components separately if need be, we may assume that our vector bundle  $\xi : E \downarrow B$  is an  $n$ -plane bundle, with projection  $p$ . The trivializations are fiberwise isomorphisms  $g_i : p^{-1}(U_i) \rightarrow \mathbb{R}^n$ . We can assemble these maps using the partition of unity, and define  $g : E \rightarrow (\mathbb{R}^n)^k$  as the unique map such that

$$\text{pr}_i g(e) = \phi_i(p(e))g_i(e).$$

This is a fiberwise linear embedding. The map  $e \mapsto (p(e), g(e))$  is an embedding into the trivial bundle  $B \times \mathbb{R}^{nk}$ .  $\square$

We can now use the standard inner product on  $\mathbb{R}^{nk}$  (or any other metric on  $B \times \mathbb{R}^{nk}$ ) to form the complement of  $E$ :

**Corollary 19.5.** *Over a compact Hausdorff space, any vector bundle has a complement (i.e. a vector bundle  $\xi^\perp$  such that  $\xi \oplus \xi^\perp$  is trivial).*

Suppose our vector bundle has fiber dimension  $n$ . The image of  $g(E_x)$  is an  $n$ -plane in  $\mathbb{R}^{nk}$ ; that is, an element  $f(x) \in \text{Gr}_n(\mathbb{R}^{nk})$ . We have produced a diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & E(\xi_{nk,n}) \\ \xi \downarrow & & \downarrow \\ B & \xrightarrow{f} & \text{Gr}_n(\mathbb{R}^{nk}) \end{array}$$

that expresses  $\xi$  as the pullback of the tautologous bundle  $\xi_{nk,n}$  under a map  $f : B \rightarrow \text{Gr}_n(\mathbb{R}^{nk})$ . This map  $f$ , covered by a bundle map, is a *Gauss map* for  $\xi$ .

### The Grassmannian model

The frame bundle of the tautologous vector bundle over the Grassmannian  $\text{Gr}_n(\mathbb{C}^{n+k})$  is the complex Stiefel manifold

$$V_n(\mathbb{C}^{n+k}) = \{\text{isometric embeddings } \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+k}\}.$$

Ehresmann’s Theorem 4.5 (for example) tells us that the projection map

$$V_n(\mathbb{C}^{n+k}) \downarrow \text{Gr}_n(\mathbb{C}^{n+k})$$

sending an embedding to its image is a fiber bundle, so we have a principal  $U(n)$ -bundle.

How connected is this complex Stiefel variety?  $U(q)$  acts transitively on the unit sphere in  $\mathbb{C}^q$  and the isotropy group of the basis vector  $e_q$  is  $U(q-1)$  embedded in  $U(q)$  in the upper left corner. So we get a tower of fiber bundles with the indicated fibers:

$$\begin{array}{ccc} S^{2k+1} & \longrightarrow & U(n+k)/U(k) & = & V_n(\mathbb{C}^{n+k}) \\ & & \downarrow & & \\ S^{2k+3} & \longrightarrow & U(n+k)/U(1+k) & & \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ S^{2(n+k)-1} & \xrightarrow{=} & U(n+k)/U((n-1)+k). & & \end{array}$$

The long exact homotopy sequence shows that  $V_n(\mathbb{C}^{n+k})$  is  $(2k)$ -connected. It’s a “twisted product” of the the spheres  $S^{2k+1}, S^{2k+3}, \dots, S^{2(n+k)-1}$ .

So forming the direct limit

$$V_n(\mathbb{C}^\infty) = \lim_{k \rightarrow \infty} V_n(\mathbb{C}^{n+k})$$

gives us a contractible CW complex with a principal action of  $U(n)$ . The quotient map

$$V_n(\mathbb{C}^\infty) \downarrow V_n(\mathbb{C}^\infty)/U(n) = \text{Gr}_k(\mathbb{C}^\infty)$$

provides us with a universal principal  $U(n)$  bundle, and hence also a universal  $n$ -plane bundle  $\xi_n$ . An element of  $E(\xi_n)$  is an  $n$ -dimensional subspace of the countably infinite dimensional vector space  $\mathbb{C}^\infty$ . This is the “infinite Grassmannian,” and it deserves the symbol  $BU(n)$ .

Dividing by a closed subgroup  $G \subseteq U(n)$  provides us with a model for  $BG$ . Of course sometimes we have more direct constructions; for example the same observations show that  $BO(n)$  is the space of  $n$ -planes in  $\mathbb{R}^\infty$ .

## 20 Simplicial sets and classifying spaces

We encountered simplicial sets at the very beginning of 18.905, as a step on the way to constructing singular homology. We will take this story up again here, briefly, because simplicial methods provide a way to organize the combinatorial data needed for the construction of classifying spaces and maps. It turns out that simplicial sets actually afford a completely combinatorial model for homotopy theory, though that is a story for another time.

### Simplex category and nerve

The *simplex category*  $\Delta$  has as objects the finite totally ordered sets

$$[n] = \{0, 1, \dots, n\}, n \geq 0,$$

and as morphisms the order preserving maps. In particular the “coface” map  $d^i : [n] \rightarrow [n+1]$  is injection omitting  $i$  and the “codegeneracy” map  $s^i : [n] \rightarrow [n-1]$  is the surjection repeating  $i$ . Any order-preserving map can be written as the composite of these maps, and there are famous relations that they satisfy. They generate the category  $\Delta$ .

The *standard (topological) simplex* is the functor  $\Delta : \Delta \rightarrow \mathbf{Top}$  defined by sending  $[n]$  to the “standard  $n$ -simplex”  $\Delta^n$ , the convex hull of the standard basis vectors  $e_0, e_1, \dots, e_n$  in  $\mathbb{R}^{n+1}$ . Order-preserving maps get sent to the affine extension of the map on basis vectors. So  $d^i$  includes the  $i$ th codimension 1 face, and  $s^i$  collapses onto a codimension 1 face.

**Definition 20.1.** Let  $\mathcal{C}$  be a category. Denote by  $s\mathcal{C}$  the category of *simplicial objects* in  $\mathcal{C}$ , i.e., the category  $\text{Fun}(\Delta^{op}, \mathcal{C})$ . We write  $X_n = X([n])$  for the “object of  $n$ -simplices.”

A simplicial object can be defined by giving an object  $X_n \in \mathcal{C}$  for every  $n \geq 0$  along with maps  $d_i : X_{n+1} \rightarrow X_n$  and  $s_i : X_{n-1} \rightarrow X_n$  satisfying certain quadratic identities.

Our first example of a simplicial object is the *singular simplicial set*  $\text{Sin}(X)$  of a space  $X$ :

$$\text{Sin}(X)_n = \text{Sin}_n(X) = \mathbf{Top}(\Delta^n, X).$$

There is a categorical analogue of  $\Delta : \Delta \rightarrow \mathbf{Top}$ . After all, the ordered set  $[n]$  is a particularly simple small category:  $\Delta$  is a full subcategory of the category of small categories. So a small category  $C$  determines a simplicial set  $NC$ , the *nerve* of  $C$ , with

$$(NC)_n = N_n C = \text{Fun}([n], C).$$

Thus  $N_0 C$  is the set of objects of  $C$ ;  $N_1 C$  is the set of morphisms;  $d_0 : N_1 C \rightarrow N_0 C$  sends a morphism to its target, and  $d_1 : N_1 C \rightarrow N_0 C$  sends a morphism to its source;  $s_0 : N_0 C \rightarrow N_1 C$

sends an object to its identity morphism. In general  $N_n C$  is the set of  $n$ -chains in  $C$ : composable sequences of  $n$  morphisms. For  $0 < i < n$ , the face map  $d_i : N_n C \rightarrow N_{n-1} C$  forms the composite of two adjacent morphisms, while  $d_0$  omits the initial morphism and  $d_n$  omits the terminal morphism. Degeneracies interpose identity maps.

For example, a group  $G$  can be regarded as a small category, one with just one object. We denote it again by  $G$ . Then  $N_n G = G^n$ , and for  $0 < i < n$

$$d_i(g_1, \dots, g_n) = (g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n).$$

while

$$d_0(g_1, \dots, g_n) = (g_2, \dots, g_n), \quad d_n(g_1, \dots, g_n) = (g_1, \dots, g_{n-1}).$$

In general, the nerve construction allows us to regard small categories as a special class of simplicial sets. This attitude is the starting point for the theory of “quasi-categories” or “ $\infty$ -categories,” which constitute a somewhat more general class of simplicial sets.

## Realization

The functor  $\text{Sin}$  transported us from spaces to simplicial sets. Milnor [24] described how to go the other way.

Let  $K$  be a simplicial set. The *geometric realization*  $|K|$  of  $K$  is

$$|K| = \left( \prod_{n \geq 0} \Delta^n \times K_n \right) / \sim$$

where  $\sim$  is the equivalence relation defined by:

$$\Delta^m \times K_m \ni (v, \phi^* x) \sim (\phi_* v, x) \in \Delta^n \times K_n$$

for all maps  $\phi : [m] \rightarrow [n]$ .

**Example 20.2.** The equivalence relation is telling us to glue together simplices as dictated by the simplicial structure on  $K$ . To see this in action, let us look at  $\phi^* = d_i : K_{n+1} \rightarrow K_n$  and  $\phi_* = d^i : \Delta^n \rightarrow \Delta^{n+1}$ . In this case, the equivalence relation then says that  $(v, d_i x) \in \Delta^n \times K_n$  is equivalent to  $(d^i v, x) \in \Delta^{n+1} \times K_{n+1}$ . In other words: the  $i$ th face of the  $n+1$  simplex labeled by  $x$  is identified with the  $n$ -simplex labeled by  $d_i x$ .

There’s a similar picture for the degeneracies  $s^i$ , where the equivalence relation dictates that every element of the form  $(v, s_i x)$  is already represented by a simplex of lower dimension. A simplex in a simplicial set is “nondegenerate” if it is not in the image of a degeneracy map. Neglecting the topology,  $|X|$  is the disjoint union of (topological) simplex interiors labeled by the nondegenerate simplices of  $K$ .

**Example 20.3.** Let  $n \geq 0$ , and consider the simplicial set  $\mathbf{\Delta}(-, [n])$ . This is called the “simplicial  $n$ -simplex”, for good reason: Its geometric realization is canonically homeomorphic to the geometric  $n$ -simplex  $\Delta^n$ .

The realization  $|K|$  of a simplicial set has a naturally defined CW structure with

$$\text{sk}_n |K| = \left( \prod_{k \leq n} \Delta^k \times K_k \right) / \sim .$$

The face maps give the attaching maps; for more details, see [11, Proposition I.2.3]. This is a very combinatorial way to produce CW-complexes.

The geometric realization functor and the singular simplicial set functor form one of the most important and characteristic examples of an adjoint pair:

$$|-| : s\mathbf{Set} \rightleftarrows \mathbf{Top} : \text{Sin}$$

The adjunction morphisms are easy to describe. For  $K \in s\mathbf{Set}$ , the unit for the adjunction  $K \rightarrow \text{Sin}|K|$  sends  $x \in K_n$  to the map  $\Delta^n \rightarrow |K|$  defined by  $v \mapsto [(v, x)]$ .

To describe the counit, let  $X$  be a space. There is a continuous map  $\Delta^n \times \text{Sin}_n(X) \rightarrow X$  given by  $(v, \sigma) \mapsto \sigma(v)$ . The equivalence relation defining  $|\text{Sin}(X)|$  says precisely that the map factors through the dotted map in the following diagram:

$$\begin{array}{ccc} |\text{Sin}(X)| & \overset{\text{---}}{\dashrightarrow} & X \\ \uparrow & \nearrow & \\ \coprod \Delta^n \times \text{Sin}_n(X) & & \end{array}$$

A theorem of Milnor [24] asserts that this map is a weak equivalence. This provides a functorial (and therefore spectacularly inefficient) CW approximation for any space.

This adjoint pair enjoys properties permitting the wholesale comparison of the homotopy theory of spaces with a combinatorially defined homotopy theory of simplicial sets. For more details, see for example [11].

### Classifying spaces

Combining the two constructions we have just discussed, we can assign to any small category  $C$  a space

$$BC = |NC|,$$

known as its *classifying space*. For example,  $B[n] = \Delta^n$ .

When  $C$  is a group,  $G$ , this space does in fact support a principal  $G$ -bundle. Before we explain that, let's look at the example of the group  $C_2$  of order 2. Write  $t$  for the non-identity element of  $C_2$ . There is just one non-degenerate  $n$  simplex in  $NC_2$  for any  $n \geq 0$ , namely  $(t, t, \dots, t)$ . So the realization  $BC_2$  has a single  $n$ -cell for every  $n$ . Not bad, since it's supposed to be a CW structure on  $\mathbb{R}P^\infty$ ! Think about what the low skeletons are. There's just one object, so  $(BC_2)_0 = *$ . There is just one nondegenerate 1-simplex,  $(t) \in C_2^1$ , so  $(BC_2)_1$  is a circle. There's just one nondegenerate 2-simplex,  $(t, t) \in C_2^2$ . Its faces are

$$d_0(t, t) = t, \quad d_1(t, t) = t^2 = 1, \quad d_2(t, t) = t.$$

The middle face has been identified with  $*$  since it's degenerate, and we see a standard representation of  $\mathbb{R}P^2$  as a "lune" with its two edges identified. A similar analysis shows that  $(BC_2)_n = \mathbb{R}P^n$  for any  $n$ .

The projection maps  $C \times D \rightarrow C$  and  $C \times D \rightarrow D$  together induce a natural map

$$B(C \times D) \rightarrow BC \times BD.$$

**Lemma 20.4.** *The classifying space construction sends natural transformations to homotopies.*

*Proof.* A natural transformation of functors  $C \rightarrow D$  is the same thing as a functor  $C \times [1] \rightarrow D$ . Since  $B[1] = \Delta^1$ , we can form the homotopy

$$BC \times \Delta^1 = BC \times B[1] \rightarrow B(C \times [1]) \rightarrow BD$$

□

**Corollary 20.5.** *An adjoint pair induces a homotopy equivalence on classifying spaces.*

**Corollary 20.6.** *If  $C$  contains an initial object or a terminal object then  $BC$  is contractible.*

*Proof.* Saying that  $o \in C$  is initial is saying that the inclusion  $o : [0] \rightarrow C$  is a left adjoint. □

The following is a nice surprise, and requires the use of the compactly generated topology on the product.

**Theorem 20.7.** *The natural map  $B(C \times D) \rightarrow BC \times BD$  is a homeomorphism.*

*Sketch of proof.* This is nontrivial – not “categorical” – because it asserts that certain limits commute with certain colimits. The underlying fact is the Eilenberg-Zilber theorem, which gives a simplicial decomposition of  $\Delta^m \times \Delta^n$  and verifies the result when  $C = [m]$  and  $D = [n]$ . The general result follows since every simplicial set is a colimit of its “diagram of simplices,” and  $B$  respects colimits. □

## The translation groupoid

An action of  $G$  on a set  $X$  determines a category, a groupoid in fact, the “translation groupoid,” which I will denote by  $GX$ . Its object set is  $X$ , and

$$GX(x, y) = \{g \in G : gx = y\}$$

Composition comes from the group multiplication. This is a special case of the “Grothendieck construction.”

When  $X = *$  we recover the category  $G$ . Another case of interest is when  $X = G$  with  $G$  acting from the left by translation. The category  $GG$  is “unicursal”: there is exactly one morphism between any two objects; every object is both initial and terminal. This implies that  $B(GG)$  is contractible.

The association

$$X \mapsto GX \mapsto N(GX) \mapsto |N(GX)| = B(GX)$$

is functorial. In particular, right multiplication by  $g \in G$  on the set  $G$  is equivariant with respect to the left action of  $G$  on it. Therefore  $G$  acts from the right on  $GG$  and hence on  $B(GG)$ . This is a “free” action: no  $g \in G$  except the identity element fixes any simplex. This implies that  $B(GG)$  admits the structure of a free  $G$ -CW complex. It’s not hard to verify that  $B(GG)/G = BG$ , so we have succeeded in constructing a functorial classifying space for any discrete group.

## 21 The Čech category and classifying maps

In this lecture I’ll sketch a program due to Graeme Segal [33] (1941–, Oxford) for classifying principal  $G$ -bundles using the simplicial description of the classifying space proposed in the last lecture. That machinery admits an extension to general topological groups.

## Top-enrichment

The Grassmannian model provides a classifying space for any compact Lie group. This includes finite discrete groups, which are also covered by the construction we just did. But we'd like to provide a construction to cover arbitrary topological groups.

**Definition 21.1.** A category *enriched in Top* is a category  $\mathcal{C}$  together with topologies on all the morphism sets, with the property that the composition maps are continuous.

The fact that **Top** is Cartesian closed provides us with an enrichment in **Top** of the category **Top** itself. A simpler (and smaller) example is given by any topological group (or monoid), regarded as a category with one object. Then a continuous action of  $G$  on a space  $X$  is just a functor  $G \rightarrow \mathbf{Top}$  that is continuous on hom spaces: a “topological functor.”

The “nerve” construction now produces a simplicial *space*,

$$NG \in s\mathbf{Top}$$

associated to any topological group  $G$ . The formula for geometric realization still makes perfectly good sense for a simplicial space. (It won't generally be a CW complex anymore, but it does have a useful “skeleton” filtration given by assembling only simplices of dimension up to  $n$ .) Combining the two constructions, we may form the “classifying space”

$$BG = |NG|.$$

This provides a functorially defined classifying space for topological groups.

## Internal categories

To justify this language, we should produce a principal  $G$ -bundle over this space with contractible total space. This construction requires one further invasion of topology into category theory (or vice versa), namely, an “internal category” in **Top**.

**Definition 21.2.** *Top-category* is a pair of spaces  $C_0$  and  $C_1$  (to be thought of as the space of objects and the space of morphisms), together with continuous structure maps

$$\text{source, target} : C_1 \rightrightarrows C_0, \quad \text{identity} : C_0 \rightarrow C_1$$

$$\text{composition} : C_1 \times_{C_0} C_1 \rightarrow C_1$$

satisfying the axioms of a category.

If the object space is discrete, this is just an enrichment in **Top**. But there are other important examples. The simplest one is entirely determined by a space  $X$ : write  $cX$  for it. Just take it  $(cX)_0 = (cX)_1 = X$  with the “identity” map  $(cX)_0 \rightarrow (cX)_1$  given by the identity map.

The nerve and classifying space constructions carry over without change to this new setting.  $(NC)_0$  will no longer be discrete. The classifying space of  $cX$  is just  $X$ , for example. The observation that an adjoint pair yields a homotopy equivalence still holds.

Now suppose that  $G$  acts on a space  $X$ . The construction of  $GX$  carried out in the previous lecture provides us with a **Top**-category. Its classifying space maps to that of  $G$ , since  $X$  maps to a point.

**Proposition 21.3.** *If  $G$  is a Lie group (and much more generally as well) the map  $B(GG) \rightarrow BG$  is a principal  $G$ -bundle, and  $B(GG)$  is contractible.*

So this gives the classifying space of  $G$ , functorially in  $G$ . It's not hard to see that in fact

$$B(GX) = B(GG) \times_G X.$$

This degree of generality provides an inductive way to construct Eilenberg Mac Lane spaces explicitly. Begin with any discrete abelian group  $\pi$ . Apply the classifying space construction we've just described, to obtain a  $K(\pi, 1)$ . Now being abelian is equivalent to the multiplication map  $\pi \times \pi \rightarrow \pi$  being a homomorphism. So we may leverage the functoriality of  $B$ , and the fact that it commutes with products, and form

$$B\pi \times B\pi \cong B(\pi \times \pi) \rightarrow B\pi.$$

This provides on  $B\pi$  the structure of a topological abelian group. So we can apply  $B$  again:  $BB\pi = K(\pi, 2)$ . And so on:

$$B^n \pi = K(\pi, n).$$

## Descent

Let  $\pi : Y \rightarrow X$  be a map of spaces. We can use it to define a **Top**-category, the “descent category” or “Čech category”  $\check{C}(\pi)$ , as follows. The space of objects is  $X$ , and the space of morphisms is  $Y \times_X Y$ . The structure maps are given by

$$\begin{aligned} \text{id} = \Delta : Y &\rightarrow Y \times_X Y & y &\mapsto (y, y) \\ \text{source} = \text{pr}_1 : Y \times_X Y &\rightarrow Y & (y_1, y_2) &\rightarrow y_1 \\ \text{target} = \text{pr}_2 : Y \times_X Y &\rightarrow Y & (y_1, y_2) &\rightarrow y_2 \\ \text{composition} : (Y \times_X Y) \times_Y (Y \times_X Y) &\rightarrow Y \times_X Y & ((y_1, y_2), (y_2, y_3)) &\mapsto (y_1, y_3). \end{aligned}$$

There is a continuous functor

$$\tilde{\pi} : \check{C}(\pi) \rightarrow cX$$

determined by mapping the object space by the identity.

This construction is best understood from its motivating case. Suppose that  $\mathcal{U}$  is a cover of  $X$  and let

$$Y = \coprod_{U \in \mathcal{U}} U,$$

mapping to  $X$  by sending  $x \in U$  to  $x \in X$ . Then

$$Y \times_X Y = \coprod_{(U, V) \in \mathcal{U} \times \mathcal{U}} U \cap V,$$

the disjoint union of intersections of ordered pairs of elements of  $\mathcal{U}$ . Source and target just embed  $U \cap V$  into  $U$  and  $V$ .

In this case let's write  $\check{C}(\mathcal{U})$  for the Čech category. In good cases we can recover  $X$  from  $\check{C}(\mathcal{U})$ :

**Proposition 21.4.** *If the open cover  $\mathcal{U}$  of  $X$  admits a subordinate partition of unity, then  $B\tilde{\pi} : B\check{C}(\mathcal{U}) \rightarrow X$  is a homotopy equivalence.*

*Proof.* A sequence  $U_0, U_1, \dots, U_n$  of elements of  $\mathcal{U}$  together with a point  $x$  in their intersection determines a chain  $(x \in U_0) \rightarrow (x \in U_1) \rightarrow \dots \rightarrow (x \in U_n)$  in the category  $\check{C}(\mathcal{U})$ . The counit of the realization-singular adjunction then gives a map

$$\epsilon : \Delta^n \times (U_0 \cap U_1 \cap \dots \cap U_n) \rightarrow B\check{C}(\mathcal{U}).$$

Now let  $\{\phi_U : U \in \mathcal{U}\}$  be a partition of unity subordinate to  $\mathcal{U}$ , so that, for every  $x \in X$ ,  $\phi_U(x) = 0$  for all but finitely many  $U \in \mathcal{U}$ , and  $\sum_U \phi_U = 1$ . Pick a partial order on the elements of  $\mathcal{U}$  that is total on any subset with nonempty intersection. For any  $x$  let  $U_0(x), \dots, U_{n(x)}(x)$  be the ordered sequence of elements of  $\mathcal{U}$  that contain  $x$ . Then define

$$X \rightarrow B\check{C}(\mathcal{U})$$

by sending

$$x \mapsto \epsilon((\phi_{U_0(x)}(x), \dots, \phi_{U_{n(x)}(x)}(x)), x).$$

It's not hard to check that this gives a well-defined map that is homotopy inverse to  $B\tilde{\pi}$ .  $\square$

**Remark 21.5.** A final comment: In [33] Segal explains how to use these methods to construct a spectral sequence from this approach, one that includes the Serre spectral sequence and more generally the topological version of the Leray spectral sequence. We won't pursue that avenue in these lectures, though, but instead will describe two other approaches.

### Transition functions, cocycles, and classifying maps

Now suppose that  $p : P \rightarrow B$  is a principal  $G$ -bundle. Pick a trivializing open cover  $\mathcal{U}$ , along with trivializations  $\varphi_U : p^{-1}U \rightarrow U \times G$  for  $U \in \mathcal{U}$ . These data determine a continuous functor

$$\check{C}(\mathcal{U}) \rightarrow G$$

as follows. There's no choice about behavior on objects. On morphisms, we use the "transition functions" associated with the given trivializations. So for  $U, V \in \mathcal{U}$ , the intersection  $U \cap V$  is a subspace of the space of morphisms in  $\check{C}(\mathcal{U})$ . We map it to  $G$  by

$$x \mapsto \varphi_V(x)\varphi_U(x)^{-1} \in G.$$

The "cocycle condition" on these transition functions is the statement that together these maps constitute a functor.

Therefore we get a diagram

$$\begin{array}{ccc} B\check{C}(\mathcal{U}) & \longrightarrow & BG \\ \uparrow \simeq & & \\ X & & \end{array}$$

and one can check that the bundle  $EG \downarrow BG$  pulls back to  $P \downarrow X$  under the composite  $X \rightarrow BG$ .

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