MULTIPARTICLE STATES AND TENSOR PRODUCTS

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1 Introduction to the Tensor Product

In this section, we develop the tools needed to describe a system that contains more than one particle. Most of the required ideas appear when we consider systems with two particles. We will assume the particles are distinguishable; for indistinguishable particles quantum mechanics imposes some additional constraints on the allowed set of states. We will study those constraints later in the course (or in 8.06!) The tools we are about to develop will be needed to understand addition of angular momenta. In that problem one is adding the angular momenta of the two or more particles in the system.

Consider then two particles. Below is a description of the quantum mechanics and family of operators associated with each particle:

- Particle 1: its quantum mechanics is described by a complex vector space V. It has associated operators $T_1, T_2, ...$
- Particle 2: its quantum mechanics is described by a complex vector space W. It has associated operators $S_1, S_2, ...$

This list of operators for each particle may include some or many of the operators you are already familiar with: position, momentum, spin, Hamiltonians, projectors, etc.

Once we have two particles, the two of them together form our system. We are after the description of quantum states of this two-particle system. On first thought, we may think that any state of this system should be described by giving the state $v \in V$ of the first particle and the state $w \in W$ of the second particle. This information could be represented by the ordered list (v, w) where the first item

is the state of the first particle and the second item the state of the second particle. This *is* a state of the two-particle system, but it is far from being the general state of the two-particle system. It misses remarkable new possibilities, as we shall soon see.

We thus introduce a new notation. Instead of representing the state of the two-particle system with particle one in v and particle two in w as (v, w), we will represent it as $v \otimes w$. This element $v \otimes w$ will be viewed as a vector in a new vector space $V \otimes W$ that will carry the description of the quantum states of the system of two particles. This $\otimes$ operation is called the "tensor product." In this case we have two vector spaces over $\mathbb{C}$ and the tensor product $V \otimes W$ is a new complex vector space:

$$v \otimes w \ \in \ V \otimes W \quad \text{when} \quad v \in V,\ w \in W\,. \tag{1.1}$$

In $v \otimes w$ there is no multiplication to be carried out, we are just placing one vector to the left of $\otimes$ and another to the right of $\otimes$.

We have only described some elements of $V \otimes W$, not quite given its definition yet.[1] We now explain two physically motivated rules that define the tensor product completely.

1. If the vector representing the state of the first particle is scaled by a complex number this is equivalent to scaling the state of the two particles. The same for the second particle. So we declare

$$\boxed{(av) \otimes w \ = \ v \otimes (aw) \ = \ a\ (v \otimes w), \qquad a \in \mathbb{C}\,.} \tag{1.2}$$

2. If the state of the first particle is a superposition of two states, the state of the two-particle system is also a superposition. We thus demand distributive properties for the tensor product:

$$\boxed{\begin{aligned} (v_1 + v_2) \otimes w \ &= v_1 \otimes w + v_2 \otimes w\,, \\ v \otimes (w_1 + w_2) \ &= v \otimes w_1 + v \otimes w_2\,. \end{aligned}} \tag{1.3}$$

The tensor product $V \otimes W$ is thus defined to be the vector space whose elements are (complex) linear combinations of elements of the form $v \otimes w$, with $v \in V, w \in W$, with the above rules for manipulation. The tensor product $V \otimes W$ is the complex vector space of states of the two-particle system!

Comments

1. The vector $0 \in V \otimes W$ is equal to $0 \otimes w$ or $v \otimes 0$. Indeed, by the first property above, with $a = 0$, we have $av = 0$ (rhs a vector) and $0 \otimes w = 0(0 \otimes w) = 0$

2. Let $v_1, v_2 \in V$ and $w_1, w_2 \in W$. A vector in $V \otimes W$ constructed by superposition is

$$\alpha_1(v_1 \otimes w_1) + \alpha_2(v_2 \otimes w_2) \in V \otimes W \tag{1.4}$$

[1] If we just left it like this, we would have defined the direct product of vector spaces.

This shows clearly that a general state of the two-particle system cannot be described by stating the state of the first particle and the state of the second particle. The above superpositions give rise to entangled states. An entangled state of the two particles is one that, roughly, cannot be disentangled into separate states of each of the particles. We will make this precise soon.

If $(e_1, \ldots, e_n)$ is a basis of V and $(f_1, \ldots, f_m)$ is a basis of W, then the set of elements $e_i \otimes f_j$ where $i = 1, \ldots, n$ and $f = 1, \ldots, m$ forms a basis for $V \otimes W$. It is simple to see these span the space since for any $v \otimes w$ we have $v = \sum_i v_i e_i$ and $w = \sum_j w_j f_j$ so that

$$v \otimes w \;=\; \big(\sum_i v_i e_i\big) \otimes \big(\sum_j w_j f_j\big) \;=\; \sum_{i,j} v_i w_j \, e_i \otimes f_j \,. \tag{1.5}$$

Given this, we see that the basis also spans linear superpositions of elements of the form $v \otimes w$, thus general elements of $V \otimes W$. With $n \cdot m$ basis vectors, the dimensionality of $V \otimes W$ is equal to the *product* of the dimensionalities of V and W:

$$\dim(V \otimes W) = \dim(V) \times \dim(W) \,. \tag{1.6}$$

Dimensions are multiplied (not added) in a tensor product.

How do we construct operators that act in the vector space $V \otimes W$? Let T be an operator in V and S be an operator in W. In other words, $T \in \mathcal{L}(V)$ and $S \in \mathcal{L}(W)$. We can then construct an operator $T \otimes S$

$$T \otimes S \;\in\; \mathcal{L}(V \otimes W) \tag{1.7}$$

defined to act as follows:

$$T \otimes S \;(v \otimes w) \;\equiv\; Tv \otimes \, Sw \,. \tag{1.8}$$

This is the only 'natural' option: we let T act on the vector it knows how to act, and S act on the vector it knows how to act.

Suppose that we want the operator $T \in \mathcal{L}(V)$ that acts on the first particle to act on the tensor product $V \otimes W$, even though we have not supplied an operator S to act on the W part. For this we upgrade the operator from one that acts on a single vector space to one, given by $T \otimes 1$, that acts on the tensor product:

$$T \in \mathcal{L}(V) \quad \to \quad T \otimes \mathbf{1} \in \mathcal{L}(V \otimes W) \,, \qquad T \otimes \mathbf{1} \,(v \otimes w) \equiv Tv \otimes w \,. \tag{1.9}$$

Similarly, an operator S belonging to $\mathcal{L}(W)$ is upgraded to $\mathbf{1} \otimes S$ to act on the tensor product. A basic result is that upgraded operators of the first particle **commute** with upgraded operators of the second particle. Indeed,

$$\begin{aligned} (T \otimes \mathbf{1}) \cdot (\mathbf{1} \otimes S) \;(v \otimes w) \;&=\; (T \otimes \mathbf{1})(v \otimes Sw) \;=\; Tv \otimes Sw \\ (1 \otimes S) \cdot (T \otimes \mathbf{1}) \;(v \otimes w) \;&= (\mathbf{1} \otimes S) \;(Tv \otimes w) \;=\; Tv \otimes Sw \,. \end{aligned} \tag{1.10}$$

and therefore

$$\big[\, T \otimes \mathbf{1}\,,\, \mathbf{1} \otimes S \,\big] \;=\; 0\,. \tag{1.11}$$

Given a system of two particles we can construct a simple total Hamiltonian H_T (describing no interactions) by upgrading each of the Hamiltonians H_1 and H_2 and adding them:

$$H_T \;\equiv\; H_1 \otimes 1 + 1 \otimes H_2 \tag{1.12}$$

Exercise. Convince yourself that

$$\exp\Big(-\frac{iH_T t}{\hbar}\Big) \;=\; \exp\Big(-\frac{iH_1 t}{\hbar}\Big) \otimes \exp\Big(-\frac{iH_2 t}{\hbar}\Big) \tag{1.13}$$

We turn now to a famous example at the basis of adding angular momenta.

Example 1: We have two spin-1/2 particles, and describe the first's state space V_1 with basis states $|+\rangle_1$ and $|-\rangle_1$ and the second's state space V_2 with basis states $|+\rangle_2$ and $|-\rangle_2$. The tensor product $V_1 \otimes V_2$ has four basis vectors:

$$|+\rangle_1 \otimes |+\rangle_2; \quad |+\rangle_1 \otimes |-\rangle_2; \quad |-\rangle_1 \otimes |+\rangle_2; \quad |-\rangle_1 \otimes |-\rangle_2 \tag{1.14}$$

If we follow the convention that the first ket corresponds to particle one and the second ket corresponds to particle two, the notation is simpler. The most general state of the two-particle system is a linear superposition of the four basis states:

$$|\Psi\rangle \;=\; \alpha_1|+\rangle_1 \otimes |+\rangle_2 \;+\; \alpha_2|+\rangle_1 \otimes |-\rangle_2 \;+\; \alpha_3|-\rangle_1 \otimes |+\rangle_2 \;+\; \alpha_4|-\rangle_1 \otimes |-\rangle_2\,. \tag{1.15}$$

Example 2: We now want to act on this state with the *total* z-component of angular momentum. Naively, this would be the sum of the z-components of each individual particle. However, we know better at this point - summing the two angular momenta really means constructing a new operator in the tensor product vector space:

$$S_z^T \;=\; S_z^{(1)} \otimes \mathbf{1} \;+\; \mathbf{1} \otimes S_z^{(2)}\,. \tag{1.16}$$

Performing the calculation in two parts,

$$\begin{aligned}
(S_z^{(1)} \otimes \mathbf{1})|\Psi\rangle \;&=\; \alpha_1 S_z|+\rangle_1 \otimes |+\rangle_2 \,+\, \alpha_2 S_z|+\rangle_1 \otimes |-\rangle_2 \,+\, \alpha_3 S_z|-\rangle_1 \otimes |+\rangle_2 \,+\, \alpha_4 S_z|-\rangle_1 \otimes |-\rangle_2 \\
&=\; \frac{\hbar}{2}\Big(\alpha_1|+\rangle_1 \otimes |+\rangle_2 \,+\, \alpha_2|+\rangle_1 \otimes |-\rangle_2 \,-\, \alpha_3|-\rangle_1 \otimes |+\rangle_2 \,-\, \alpha_4|-\rangle_1 \otimes |-\rangle_2\Big) \\
(1 \otimes S_z^{(2)})|\Psi\rangle \;&=\; \alpha_1|+\rangle_1 \otimes S_z|+\rangle_2 \,+\, \alpha_2|+\rangle_1 \otimes S_z|-\rangle_2 \,+\, \alpha_3|-\rangle_1 \otimes S_z|+\rangle_2 \,+\, \alpha_4|-\rangle_1 \otimes S_z|-\rangle_2 \\
&=\; \frac{\hbar}{2}\Big(\alpha_1|+\rangle_1 \otimes |+\rangle_2 \,-\, \alpha_2|+\rangle_1 \otimes |-\rangle_2 \,+\, \alpha_3|-\rangle_1 \otimes |+\rangle_2 \,-\, \alpha_4|-\rangle_1 \otimes |-\rangle_2\Big)
\end{aligned} \tag{1.17}$$

Adding these together, we have:

$$S_z^T|\Psi\rangle = \hbar\left(\alpha_1|+\rangle_1\otimes|+\rangle_2 \;-\; \alpha_4|-\rangle_1\otimes|-\rangle_2\right) \tag{1.18}$$

One can derive this result quickly by noting that since $S_z^{(1)}$ is diagonal in the first basis and $S_z^{(2)}$ is diagonal in the second basis, the total S_z is diagonal in the tensor space basis and its eigenvalue acting on a tensor state is the sum of the S_z eigenvalues for particle one and particle two. Thus,

$$\begin{aligned}
S_z^T|+\rangle\otimes|+\rangle &= \Big(\frac{\hbar}{2}+\frac{\hbar}{2}\Big)|+\rangle\otimes|+\rangle \;=\; \hbar|+\rangle\otimes|+\rangle \\
S_z^T|+\rangle\otimes|-\rangle &= \Big(\frac{\hbar}{2}-\frac{\hbar}{2}\Big)|+\rangle\otimes|+\rangle \;=\; 0 \\
S_z^T|-\rangle\otimes|+\rangle &= \Big(-\frac{\hbar}{2}+\frac{\hbar}{2}\Big)|+\rangle\otimes|+\rangle \;=\; 0 \\
S_z^T|-\rangle\otimes|-\rangle &= \Big(-\frac{\hbar}{2}-\frac{\hbar}{2}\Big)|-\rangle\otimes|-\rangle \;=\; -\hbar|-\rangle\otimes|-\rangle
\end{aligned} \tag{1.19}$$

The result in (1.18) follows quickly from the four relations above. Suppose we are only interested in states that have zero S_z^T. This requires

$$\alpha_1=\alpha_4=0 \quad\rightarrow\quad |\Psi\rangle=\alpha_2|+\rangle\otimes|-\rangle+\alpha_3|-\rangle\otimes|+\rangle \tag{1.20}$$

Example 3: Calculate the total x-component S_x^T of spin angular momentum on the above states with zero S_z^T. Recalling that

$$S_x|+\rangle=\frac{\hbar}{2}|-\rangle\,,\qquad S_x|-\rangle=\frac{\hbar}{2}|+\rangle \tag{1.21}$$

and writing

$$S_x^T = S_x\otimes 1 + 1\otimes S_x \tag{1.22}$$

the calculation proceeds as follows:

$$\begin{aligned}
S_x^T|+\rangle\otimes|-\rangle &= S_x|+\rangle\otimes|-\rangle \;+\; |+\rangle\otimes S_x|-\rangle \;=\; \frac{\hbar}{2}\big(|-\rangle\otimes|-\rangle \;+\; |+\rangle\otimes|+\rangle\big) \\
S_x^T|-\rangle\otimes|+\rangle &= S_x|-\rangle\otimes|+\rangle \;+\; |-\rangle\otimes S_x|+\rangle \;=\; \frac{\hbar}{2}\big(|+\rangle\otimes|+\rangle \;+\; |-\rangle\otimes|-\rangle\big)
\end{aligned} \tag{1.23}$$

Therefore

$$\begin{aligned}
S_x^T|\Psi\rangle &= \alpha_2\frac{\hbar}{2}\big(|-\rangle\otimes|-\rangle+|+\rangle\otimes|+\rangle\big)+\alpha_3\frac{\hbar}{2}\big(|+\rangle\otimes|+\rangle+|-\rangle\otimes|-\rangle\big) \\
&= \frac{\hbar}{2}(\alpha_2+\alpha_3)\big(|+\rangle\otimes|+\rangle+|-\rangle\otimes|-\rangle\big)
\end{aligned} \tag{1.24}$$

If we demand that S_x^T also be zero on the state we now find $\alpha_2=-\alpha_3$. Thus, the following state has zero S_x^T, S_z^T:

$$|\Psi\rangle \;=\; \alpha\left(|+\rangle\otimes|-\rangle-|-\rangle\otimes|+\rangle\right). \tag{1.25}$$

Exercise: Verify that $S_y^T|\Psi\rangle=0$. Thus we say that the state has total spin angular momentum zero.

We now consider the definition of an **inner product** in $V \otimes W$. To do this we simply give state how the most general inner product is computed using a basis $\{e_i \otimes f_j\}$ for the tensor product, with $\{e_i\}$ and $\{f_i\}$ *orthonormal* bases for V and W. We begin by declaring that

$$\langle e_i \otimes f_j \, , \, e_p \otimes f_q \rangle \, \equiv \, \delta_{ip}\delta_{jq} \, . \tag{1.26}$$

This makes the basis $\{e_i \otimes f_j\}$ orthonormal. In addition, we must declare that with vectors $X, Y, Z \in V \otimes W$ and a complex constant a the following axioms hold:

$$\begin{aligned} \langle X + Y \, , \, Z \rangle \; &= \; \langle X, \, Z \rangle + \langle Y, \, Z \rangle \, , \\ \langle X \, , Y + Z \rangle \; &= \; \langle X \, , Y \rangle + \langle X \, , Z \rangle \, , \\ \langle X, aY \rangle \; &= \; a \langle X, Y \rangle \\ \langle aX, Y \rangle \; &= \; a^* \langle X, Y \rangle \end{aligned} \tag{1.27}$$

This is a complete definition of the inner product in the tensor space: we can compute the inner product of any two vectors in $V \otimes W$ using the chosen basis and the above distributive rules. Indeed, using these properties we can show that

$$\langle v \otimes w \, , \, \tilde{v} \otimes \tilde{w} \rangle \, = \, \langle v \, , \, \tilde{v} \rangle \, \langle w \, , \tilde{w} \rangle \, , \tag{1.28}$$

where the inner products on the right-hand side are those in V and in W, making it clear that the inner product in $V \otimes W$ arises from the inner products in V and W. To prove this relation we begin by writing

$$\begin{aligned} v \; &= \; \sum_i v_i e_i \, , \quad w \; = \; \sum_j w_j f_j \, , \\ \tilde{v} \; &= \; \sum_p \tilde{v}_p e_p \, , \quad \tilde{w} \; = \; \sum_q \tilde{w}_q f_q \, . \end{aligned} \tag{1.29}$$

Since the basis vectors in V and W are orthonormal we find that

$$\langle v, \tilde{v} \rangle \; = \; \sum_i v_i^* \tilde{v}_i \, , \quad \langle w, \tilde{w} \rangle \; = \; \sum_j w_j^* \tilde{w}_j \, . \tag{1.30}$$

Now evaluating the left-hand side of (1.28)

$$\begin{aligned} \big\langle v \otimes w \, , \, \tilde{v} \otimes \tilde{w} \big\rangle \; &= \; \Big\langle \sum_i v_i e_i \otimes \sum_j w_j f_j \, , \, \sum_p \tilde{v}_p e_p \otimes \sum_q \tilde{w}_q f_q \Big\rangle \\ &= \; \sum_{i,j,p,q} \Big\langle v_i w_j \, e_i \otimes f_j \, , \, \tilde{v}_p \tilde{w}_q \, e_p \otimes f_q \Big\rangle \\ &= \; \sum_{i,j,p,q} v_i^* w_j^* \, \tilde{v}_p \tilde{w}_q \big\langle \, e_i \otimes f_j \, , \, e_p \otimes f_q \big\rangle \\ &= \; \sum_{i,j,p,q} v_i^* w_j^* \, \tilde{v}_p \tilde{w}_q \, \delta_{ip} \, \delta_{jq} \; = \; \sum_i v_i^* \tilde{v}_i \sum_j w_j^* \, \tilde{w}_j \\ &= \langle v, \tilde{v} \rangle \, \langle w, \, \tilde{w} \rangle \, . \end{aligned} \tag{1.31}$$

The verification that the inner-product on $V \otimes W$ satisfies the remaining axioms is left as a good practice for you. Assume below that $X, Y \in V \otimes W$. For both exercises above simply write the most general vector, as $X = \sum_{ij} x_{ij}\, e_i \otimes f_j$ and proceed.
Exercise: Show that $\langle X, X\rangle \geq 0$, and $\langle X, X\rangle = 0$ if and only if $X = 0$.
Exercise: Show that $\langle X, Y\rangle = \langle Y, X\rangle^*$.

Many times it is convenient to use bra-ket notation for inner products in the tensor product. We write

$$\begin{aligned} |v \otimes w\rangle &= |v\rangle_1 \otimes |w\rangle_2 \\ \langle v \otimes w| &= {}_1\langle v| \otimes {}_2\langle w| \,. \end{aligned} \tag{1.32}$$

Notice that both on bras and kets we write the state of particle one to the left of the state of particle two. We then write (1.28) as

$$\langle v \otimes w|\tilde{v} \otimes \tilde{w}\rangle \;=\; \big({}_1\langle v| \otimes {}_2\langle w|\big)\,\big(|\tilde{v}\rangle_1 \otimes |\tilde{w}\rangle_2\,\big) = \langle v|\tilde{v}\rangle\,\langle w|\tilde{w}\rangle\,. \tag{1.33}$$

Back to our example with spin states, our four basis vectors $|+\rangle_1 \otimes |+\rangle_2$, $|+\rangle_1 \otimes |-\rangle_2$, $|-\rangle_1 \otimes |+\rangle_2$, and $|-\rangle_1 \otimes |-\rangle_2$ are orthonormal. We had the un-normalized state in (1.25) given by

$$|\Psi\rangle = \alpha\,\Big(\,|+\rangle_1 \otimes |-\rangle_2 \;-\; |-\rangle_1 \otimes |+\rangle_2\,\Big)\,. \tag{1.34}$$

The associated bra is then

$$\langle\Psi| = \alpha^*\,\Big(\,{}_1\langle +| \otimes {}_2\langle -| \;-\; {}_1\langle -| \otimes {}_2\langle +|\,\Big)\,. \tag{1.35}$$

We then have

$$\begin{aligned} \langle\Psi|\Psi\rangle &= \alpha\alpha^*\,\Big({}_1\langle +| \otimes {}_2\langle -| \;-\; {}_1\langle -| \otimes {}_2\langle +|\Big)\Big(|+\rangle_1 \otimes |-\rangle_2 \;-\; |-\rangle_1 \otimes |+\rangle_2\,\Big) \\ &= \alpha\alpha^*\,\Big({}_1\langle +| \otimes {}_2\langle -||+\rangle_1 \otimes |-\rangle_2 \;+\; {}_1\langle -| \otimes {}_2\langle +|\,|-\rangle_1 \otimes |+\rangle_2\Big) \end{aligned} \tag{1.36}$$

since only terms where the spin states are the same for the first particle and for the second particle survive. We thus have, for normalization,

$$\langle\Psi|\Psi\rangle \;=\; |\alpha|^2(1+1) = 2|\alpha|^2 = 1\,, \quad \to \quad \alpha = \frac{1}{\sqrt{2}}\,. \tag{1.37}$$

The normalized state with zero total angular momentum is then

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\,\Big(|+\rangle_1 \otimes |-\rangle_2 - |-\rangle_1 \otimes |+\rangle_2\,\Big)\,. \tag{1.38}$$

2 Entangled States

You have learned that $V \otimes W$ includes states $\Psi = \sum_i \alpha_i \, v_i \otimes w_i$ obtained by linear superposition of simpler states of the form $v_i \otimes w_i$. If handed such a Ψ, you might want to know whether you can write it as a single term $v_* \otimes w_*$ for some $v_* \in V$ and $w_* \in W$. If so, you are able to describe the state of the particles in Ψ independently: particle one is in state v_* and particle two in state w_*. We then say that in the state Ψ the particles are *not entangled*. If no such v_* and w_* exist, we say that in the state $\Psi \in V \otimes W$ the particles are entangled or equivalently, that Ψ is an entangled stated of the two particles. Entanglement is a basis-independent property.

It is simplest to illustrate this using two-dimensional complex vector spaces V and W, like the ones we use for spin one-half. Let V have a basis e_1, e_2 and W have a basis f_1, f_2. Then, the most general state you can write is the following:

$$\Psi_A \;=\; a_{11}\, e_1 \otimes f_1 + a_{12}\, e_1 \otimes f_2 + a_{21}\, e_2 \otimes f_1 + a_{22}\, e_2 \otimes f_2 \,. \tag{2.39}$$

This state is encoded by a matrix A of coefficients

$$A \;=\; \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} . \tag{2.40}$$

The state is *not entagled* if there exist constants a_1, a_2, b_1, b_2 such that

$$a_{11}\, e_1 \otimes f_1 + a_{12}\, e_1 \otimes f_2 + a_{21}\, e_2 \otimes f_1 + a_{22}\, e_2 \otimes f_2 \;=\; (a_1 e_1 + a_2 e_2) \otimes (b_1 f_1 + b_2 f_2) \,. \tag{2.41}$$

Note that these four unknown constants are not uniquely determined: we can, for example, multiply a_1 and a_2 by some constant $c \neq 0$ and divide b_1 and b_2 by c, to obtain a different solution. Indeed $v \otimes w = (cv) \otimes (w/c)$ for any $c \neq 0$. Using the distributive laws for $\otimes$ to expand the right-hand side of (2.41) and recalling that $e_i \otimes f_j$ are basis vectors in the tensor product, we see that the equality requires the following four relations:

$$\begin{aligned} a_{11} &= a_1 b_1 \\ a_{12} &= a_1 b_2 \\ a_{21} &= a_2 b_1 \\ a_{22} &= a_2 b_2 \end{aligned} \tag{2.42}$$

Combining these four expressions leaves us with a consistency condition:

$$a_{11}a_{22} - a_{12}a_{21} = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1 \;=\; 0 \quad \rightarrow \quad \det A = 0 \,. \tag{2.43}$$

In other words, if Ψ_A is not entangled the determinant of the matrix A must be zero. We can in fact show that $\det A = 0$ implies that Ψ_A is not entangled. To do this we simply have to present a solution for the equations above under the condition $\det A = 0$.

Assume first that $a_{11} = 0$. Then $\det A = 0$ implies $a_{12}a_{21} = 0$. If $a_{12} = 0$ then

$$\Psi_A = a_{21} e_2 \otimes f_1 + a_{22} e_2 \otimes f_2 = e_2 \otimes (a_{21} f_1 + a_{22} f_2) \tag{2.44}$$

and the state is indeed not entangled. If $a_{21} = 0$ then

$$\Psi_A = a_{12} e_1 \otimes f_2 + a_{22} e_2 \otimes f_2 = (a_{12} e_1 + a_{22} e_2) \otimes f_2 \tag{2.45}$$

and again, the state is not entangled. Thus, we can solve all equations when $a_{11} = 0$. Now assuming $a_{11} \neq 0$ we can take

$$a_1 = \sqrt{a_{11}}\,, \qquad b_1 = \sqrt{a_{11}}\,, \tag{2.46}$$

to solve the first equation in (2.42). The second and third equations allow us to solve for b_2 and a_2

$$b_2 = \frac{a_{12}}{\sqrt{a_{11}}}\,, \quad a_2 = \frac{a_{21}}{\sqrt{a_{11}}} \tag{2.47}$$

The fourth equation is then automatically satisfied as

$$a_2 b_2 \;=\; \frac{a_{12}a_{21}}{a_{11}} \;=\; \frac{a_{11}a_{22}}{a_{11}} \;=\; a_{22} \tag{2.48}$$

using the vanishing determinant condition. We have thus solved the system of equations and we can write

$$\Psi_A \;=\; \left(\sqrt{a_{11}} e_1 + \frac{a_{21}}{\sqrt{a_{11}}} e_2\right) \otimes \left(\sqrt{a_{11}} f_1 + \frac{a_{12}}{\sqrt{a_{11}}} f_2\right) \quad \text{if} \quad \det A = 0\,. \tag{2.49}$$

We have thus proved that Ψ_A is entangled if and only if $\det A \neq 0$. For vector spaces of dimensions different than two the conditions for entanglement take a different form. Schrödinger called "entanglement" the essential feature of quantum mechanics.

Example: Consider our state of zero total spin angular momentum:

$$|\Phi\rangle_A \equiv \frac{1}{\sqrt{2}}\Big(|+\rangle_1 \otimes |+\rangle_2 \;-\; |-\rangle_1 \otimes |-\rangle_2\Big) \tag{2.50}$$

If we have the basis vectors $|e_1\rangle = |+\rangle_1, |e_2\rangle = |-\rangle_1$ and $|f_1\rangle = |+\rangle_2, |f_2\rangle = |-\rangle_2$ we see that the state is described by the matrix

$$A \;=\; \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{pmatrix} \tag{2.51}$$

Since the determinant of this matrix is not zero, the state is entangled.

3 Bell basis states

Bell states are a set of entangled basis vectors. Take $V_1 \otimes V_2$, with V_1 and V_2 both the two-dimensional complex vector space of spin-1/2 particles. For brevity of notation we will leave out the 1 and 2

subscripts on the states and the $\otimes$ in between the states; it is always understood that in $V_1 \otimes V_2$ the state of V_1 appears to the left of the state of V_2. Consider now the state

$$|\Phi_0\rangle \;\equiv\; \frac{1}{\sqrt{2}}\Big(|+\rangle|+\rangle \,+\, |-\rangle|-\rangle\Big)\,. \tag{3.52}$$

This is clearly an entangled state: its associated matrix is diagonal with equal entries of $1/\sqrt{2}$ and thus non-zero determinant. Moreover this state is unit normalized

$$\langle\Phi_0|\Phi_0\rangle = 1\,. \tag{3.53}$$

We can use this state as the first of our basis vectors for $V_1 \otimes V_2$. Since this tensor product is four-dimensional we need three more entangled basis states. Here they are:

$$|\Phi_i\rangle \;\equiv\; (\mathbf{1}\otimes\sigma_i)|\Phi_0\rangle\,,\;\; i=1,2,3. \tag{3.54}$$

We will explicitly see below that these states are entangled, but this property is clear from the definition. If $|\Psi_i\rangle$ is not entangled, it would follow that that $\mathbf{1}\otimes\sigma_i|\Psi_i\rangle$ (i not summed) is not entangled either (do you see why?). But using $\sigma_i^2 = 1$, we see that this last state is in fact $|\Phi_0\rangle$, which is entangled. This contradiction shows that $|\Phi_i\rangle$ must be entangled. It is also manifest from the definition that the $|\Phi_i\rangle$ states are unit normalized.

Let us look at the form of $|\Phi_1\rangle$:

$$\begin{aligned} |\Phi_1\rangle \;&=\; (\mathbf{1}\otimes\sigma_1)\frac{1}{\sqrt{2}}\Big(|+\rangle|+\rangle \,+\, |-\rangle|-\rangle\Big) \;=\; \frac{1}{\sqrt{2}}\Big(|+\rangle\sigma_1|+\rangle \,+\, |-\rangle\sigma_1|-\rangle\Big) \\ &=\; \frac{1}{\sqrt{2}}\Big(|+\rangle|-\rangle \,+\, |-\rangle|+\rangle\Big)\,. \end{aligned} \tag{3.55}$$

The state is clearly entangled. By analogous calculations we obtain the full list of Bell states

$$\begin{aligned} |\Phi_0\rangle \;&=\; \mathbf{1}\otimes\mathbf{1}\,|\Phi_0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|+\rangle|+\rangle + |-\rangle|-\rangle\Big) \\ |\Phi_1\rangle \;&=\; \mathbf{1}\otimes\sigma_1|\Phi_0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|+\rangle|-\rangle + |-\rangle|+\rangle\Big) \\ |\Phi_2\rangle \;&=\; \mathbf{1}\otimes\sigma_2|\Phi_0\rangle \;=\; \frac{i}{\sqrt{2}}\Big(|+\rangle|-\rangle - |-\rangle|+\rangle\Big) \\ |\Phi_3\rangle \;&=\; \mathbf{1}\otimes\sigma_3|\Phi_0\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|+\rangle|+\rangle - |-\rangle|-\rangle\Big)\,. \end{aligned} \tag{3.56}$$

By inspection we can confirm that Φ_0 is orthogonal to the other three: $\langle\Phi_0|\Phi_i\rangle = 0$. It is not much work either to see that the basis is in fact orthonormal. But a calculation is kind of fun:

$$\begin{aligned} \langle\Phi_i|\Phi_j\rangle \;&=\; \langle\Phi_0|(\mathbf{1}\otimes\sigma_i)(\mathbf{1}\otimes\sigma_j)|\Phi_0\rangle \\ &=\; \langle\Phi_0|\mathbf{1}\otimes\sigma_i\sigma_j|\Phi_0\rangle \\ &=\; \langle\Phi_0|\mathbf{1}\otimes\big(\mathbf{1}\delta_{ij} + i\epsilon_{ijk}\sigma_k\big)|\Phi_0\rangle \\ &=\; \delta_{ij}\langle\Phi_0|\mathbf{1}\otimes\mathbf{1}|\Phi_0\rangle + i\epsilon_{ijk}\langle\Phi_0|\mathbf{1}\otimes\sigma_k|\Phi_0\rangle \\ &=\; \delta_{ij}\langle\Phi_0|\Phi_0\rangle + i\epsilon_{ijk}\langle\Phi_0|\Phi_k\rangle \;=\; \delta_{ij}\,, \end{aligned} \tag{3.57}$$

as we wanted to show. Indeed, we have an orthonormal basis of entangled states.

We can solve for the old, non-entangled basis states in terms of the Bell states. We quickly find from (3.56)

$$\begin{aligned} |+\rangle|+\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle + |\Phi_3\rangle\right) \\ |-\rangle|-\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle - |\Phi_3\rangle\right) \\ |+\rangle|-\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle - i|\Phi_2\rangle\right) \\ |-\rangle|+\rangle &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle + i|\Phi_2\rangle\right). \end{aligned} \tag{3.58}$$

Introducing labels A and B for the two spaces in a tensor product $V_A \otimes V_B$ we rewrite the above equations as

$$\begin{aligned} |+\rangle_A|+\rangle_B &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle_{AB} + |\Phi_3\rangle_{AB}\right) \\ |-\rangle_A|-\rangle_B &= \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle_{AB} - |\Phi_3\rangle_{AB}\right) \\ |+\rangle_A|-\rangle_B &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle_{AB} - i|\Phi_2\rangle_{AB}\right) \\ |-\rangle_A|+\rangle_B &= \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle_{AB} + i|\Phi_2\rangle_{AB}\right), \end{aligned} \tag{3.59}$$

where $|\Phi_i\rangle_{AB}$ are the Bell states we defined above with tensor products in which the first state is in V_A and the second state is in V_B.

These basis states form the Bell basis. You could do an experiment to determine the probability of an arbitrary state being along any of the basis states in this orthonormal basis. You can use the experiment to detect which basis state the state is in. The state is, of course, a superposition of basis states, but during measurement will collapse into one of them with some probability. The Stern Gerlach device was an example of a device that allowed you to collapse a state into one basis state or another. This basis is more general, as it is not simply for two-state systems.
We conclude by presenting three facts.

1. Measuring in a basis. Given an orthonormal basis $|e_1\rangle, ..., |e_n\rangle$ we can measure a state $|\Psi\rangle$ along this basis and obtain that the probability $P(i)$ to be in the state $|i\rangle$ is $|\langle e_i|\Psi\rangle|^2$. After measurement the state will be in one of the states $|e_i\rangle$. This is exactly how it worked for the Stern-Gerlach experiment which, oriented about $\mathbf{z}$ amount to a measurement in the basis $|+\rangle, |-\rangle$.

   As another example, if we have a state with two particles A, B, we may choose the four Bell states as our orthonormal basis for the measurement. If so, after measurement the state will be in one of the Bell states $|\Phi_i\rangle_{AB}$, with probability given by the squared overlap $|\langle\Phi_i|_{AB}|\Psi\rangle|^2$.

2. Partial measurement. Suppose we have a general (entangled) state $\Psi \in V \otimes W$ of two particles. The observer Alice has access to both particles but decides to measure only the first particle along the basis $|e_1\rangle, \ldots, |e_n\rangle$ of V. How is this analyzed? As a first step we use that basis to write the state Ψ in the form

$$\Psi \;=\; \sum_i |e_i\rangle \otimes |w_i\rangle\,, \tag{3.60}$$

for some calculable vectors $|w_i\rangle$. As a second step we normalize the states $|w_i\rangle$:

$$\Psi \;=\; \sum_i \sqrt{\langle w_i|w_i\rangle}\; |e_i\rangle \otimes \frac{|w_i\rangle}{\sqrt{\langle w_i|w_i\rangle}}\,, \tag{3.61}$$

We claim that Alice will find the first particle to be in the state $|i\rangle$ with probability $\langle w_i|w_i\rangle$. After the measurement, the state of the particles will be

$$|e_i\rangle \otimes \frac{|w_i\rangle}{\sqrt{\langle w_i|w_i\rangle}}\,, \quad \text{for some value of } i\,. \tag{3.62}$$

(A justification of this answer was given in recitations.) You probably have used this rule before. As an example, suppose we have the entangled state of total spin zero:

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|+\rangle_1 \otimes |-\rangle_2 \;-\; |-\rangle_1 \otimes |+\rangle_2\Big) \tag{3.63}$$

If we measure the first particle along the $|+\rangle_1, |+\rangle_2$ basis we find

$$\begin{array}{l} \text{Probability that the first particle is in } |+\rangle = \tfrac{1}{2}\,. \;\; \text{State after measurement: } |+\rangle_1 \otimes |-\rangle_2 \\ \text{Probability that the first particle is in } |-\rangle = \tfrac{1}{2}\,. \;\; \text{State after measurement: } |-\rangle_1 \otimes |+\rangle_2 \end{array} \tag{3.64}$$

It follows that after the measurement of the first particle, a measurement of the second particle will show that its spin is always opposite to the spin of the first particle.

As a more nontrivial example, consider now the state of three particles A, B, C which live in $V_A \otimes V_B \otimes V_C$, which contains states of the type $v \otimes w \otimes u$ with $v \in V_A, w \in V_B, u \in V_C$, and their linear combinations. To analyze what happens if Alice decides to do a Bell measurement of particles A, B, the state Ψ of the system must be written in the form

$$\Psi \;=\; |\Phi_0\rangle_{AB} \otimes |u_0\rangle_C + |\Phi_1\rangle_{AB} \otimes |u_1\rangle_C + |\Phi_2\rangle_{AB} \otimes |u_2\rangle_C + |\Phi_3\rangle_{AB} \otimes |u_3\rangle_C \tag{3.65}$$

After the measurement, the state of the particles A, B will be one of the Bell states $|\Phi_\mu\rangle_{AB}$ with $\mu = 0, 1, 2, 3$. We have

$$\begin{array}{l} \text{Probability that } (A,B) \text{ is in } |\Phi_\mu\rangle_{AB} \;=\; \langle u_\mu|u_\mu\rangle\,, \\ \text{State after measurement is } |\Phi_\mu\rangle_{AB} \otimes \dfrac{|u_\mu\rangle_C}{\sqrt{\langle u_\mu|u_\mu\rangle}} \text{ for some } \mu \in \{0,1,2,3\}\,. \end{array} \tag{3.66}$$

3. The action of the Pauli matrices on spin states can be realized as time evolution via some Hamiltonian. Note first that the Pauli matrices are unitary because they are Hermitian and square to the identity. Multiplying a state by σ_1 is thus acting with a unitary operator and unitary operators generate allowed time evolution. Thus, there is a Hamiltonian that applied to a system over some length of time will turn any spin state $|\Psi\rangle$ into $\sigma_i|\Psi\rangle$. In practice, this Hamiltonian would correspond to some device with a magnetic field of some determined magnitude and direction that acts for a few picoseconds and evolves spin states in time. We can check, for example, that any Pauli matrix can be written as the exponential of i times a Hermitian matrix (which would be proportional to the Hamiltonian):

$$e^{i\frac{\pi}{2}(-1+\sigma_i)} = e^{-i\frac{\pi}{2}}e^{i\frac{\pi}{2}\sigma_i} = (-i)(i\sigma_i) = \sigma_i \tag{3.67}$$

4 Quantum Teleportation

Classically, teleportation is impossible: there is no classical basis for dematerializing an object and recreating it somewhere else. In 1993, a group of scientists (Bennet, Brassand, Crépeau, Jozsa, Peres, and Wooters) discovered that teleportation *is* possible in quantum mechanics.

Imagine that Alice has a quantum state: the state of a 1/2 particle. The state is:

$$|\Psi\rangle_C \;=\; \alpha|+\rangle_C + \beta|-\rangle_C\,, \tag{4.68}$$

where $\alpha, \beta \in \mathbb{C}$ and the letter C denotes the state space V_C of this C particle to be teleported. Her goal is to teleport this state - called a "quantum bit," or *qubit* - to Bob, who is far away.

The quantum "no-cloning" principle prevents Alice from simply creating a copy of the state and sending it to Bob. In other words, it is impossible to create a copy of a quantum mechanical state. Measuring the state and telling Bob about the result is no option either: if Alice measures the state with some Stern-Gerlach apparatus, the spin will just point up or point down. What has she learned? Almost nothing. Only with many copies of the state she would be able to learn about the values of α and β. Having just one particle she is unable to measure α and β and send those values to Bob. Of course, it may be that for some reason Alice knows the values of α and β. In some cases she could transmit that information to Bob to recreate the state. But it could also be that α, for example is some transcendental number 0.178573675623..... with no discernible rhyme or reason, and she would need an infinite amount of information to send to Bob this value.

Here is a diagram of how Alice can will teleport the information:

The key tool Alice and Bob use is an entangled states of two particles A and B, in which Alice has access to particle A and Bob has access to particle B. One pair (A, B) of entangled particles will allow Alice to teleport the state C of one particle. To teleport a full person from one place to another, we would have to have an enormous reservoir of entangled pairs, one pair needed to teleport each quantum state of particles in the body of that person. This clearly remains science-fiction.

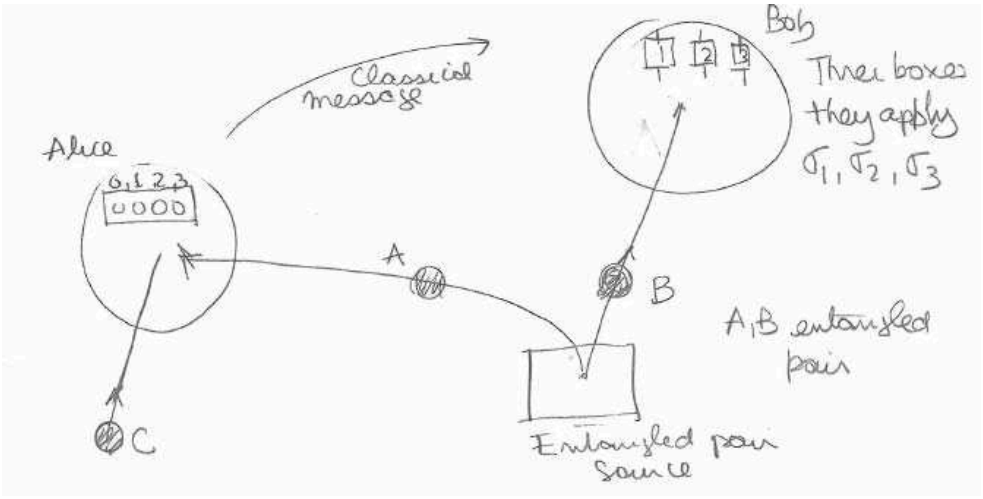

Figure 1: Alice has access to spin state C, to be teleported and to spin state A which is entangled with spin state B. Bob has spin state B. Alice will perform a Bell measurement on states A and C. After she measures, Bob's state will turn into the desired state, up to a simple unitary transformation.

Alice has a console with four lights, labeled with $\mu = 0, 1, 2, 3$. She will do a Bell measurement involving particle A (of the shared entangled pair) and particle C (the one to be teleported). As she does so, one of her four lights will blink: If it is th μ-th light it is because she ended with the Bell state $|\Phi_\mu\rangle_{AC}$. Bob, who is in possession of the particle B, has a console with four boxes which generate unitary transformations (via some Hamiltonians applied for a fixed time). The first box, labeled $\mu = 0$ does nothing to the state. The i-th box (with $i = 1, 2, 3$) applies the operator σ_i. Alice communicates to Bob that the μ-th light blinked. Then Bob submits particle B to the μ-th box and out comes, we claim, the teleported state $|\Psi\rangle_B$ as the state of particle B.

Let us prove this mathematically. Let the entangled shared pair, with A at Alice and B at Bob, be the first Bell basis state:

$$|\Phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}\big(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B\big)\,. \tag{4.69}$$

The total state of our three particles, A, B, C is therefore:

$$\begin{aligned} |\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C &= |\Phi_0\rangle_{AB} \otimes (\alpha|+\rangle_C + \beta|-\rangle_C) \\ &= \frac{1}{\sqrt{2}}\Big(|+\rangle_A|+\rangle_B + |-\rangle_A|-\rangle_B\Big) \otimes \big(\alpha|+\rangle_C + \beta|-\rangle_C\big)\,. \end{aligned} \tag{4.70}$$

Expanding out and reordering the states to have A followed by C and then by B we have

$$\begin{aligned} |\Phi_0\rangle_{AB} \otimes |\Psi\rangle_C = \frac{1}{\sqrt{2}}\Big(& \alpha \underbrace{|+\rangle_A|+\rangle_C}|+\rangle_B + \beta \underbrace{|+\rangle_A|-\rangle_C}|+\rangle_B \\ & + \alpha \underbrace{|-\rangle_A|+\rangle_C}|-\rangle_B + \beta \underbrace{|-\rangle_A|-\rangle_C}|-\rangle_B\Big)\,. \end{aligned} \tag{4.71}$$

Note that as long as we label the states, the order in which we write them does not matter! We now write these basis states with braces in the Bell basis using (3.59). We find

$$\begin{aligned} |\Phi_0\rangle_{AB}\otimes|\Psi\rangle_C \;=\; & \tfrac{1}{2}\Big(|\Phi_0\rangle_{AC}+|\Phi_3\rangle_{AC}\Big)\;\alpha|+\rangle_B+\tfrac{1}{2}\Big(|\Phi_1\rangle_{AC}-i|\Phi_2\rangle_{AC}\Big)\;\beta|+\rangle_B \\ & +\tfrac{1}{2}\Big(|\Phi_1\rangle_{AC}+i|\Phi_2\rangle_{AC}\Big)\;\alpha|-\rangle_B+\tfrac{1}{2}\Big(|\Phi_0\rangle_{AC}-|\Phi_3\rangle_{AC}\Big)\;\beta|-\rangle_B\,. \end{aligned} \tag{4.72}$$

Collecting the Bell states we find

$$\begin{aligned} |\Phi_0\rangle_{AB}\otimes|\Psi\rangle_C \;=\; & \tfrac{1}{2}|\Phi_0\rangle_{AC}\left(\alpha|+\rangle_B\;+\;\beta|-\rangle_B\right)\;+\;\tfrac{1}{2}|\Phi_1\rangle_{AC}\left(\alpha|-\rangle_B\;+\;\beta|+\rangle_B\right) \\ & +\tfrac{1}{2}|\Phi_2\rangle_{AC}\left(i\alpha|-\rangle_B\;-\;i\beta|+\rangle_B\right)\;+\;\tfrac{1}{2}|\Phi_3\rangle_{AC}\left(\alpha|+\rangle_B\;-\;\beta|-\rangle_B\right). \end{aligned} \tag{4.73}$$

We can then see that in fact we got

$$\begin{aligned} |\Phi_0\rangle_{AB}\otimes|\Psi\rangle_C \;=\; & \tfrac{1}{2}|\Phi_0\rangle_{AC}\otimes|\Psi\rangle_B\;+\;\tfrac{1}{2}|\Phi_1\rangle_{AC}\otimes\sigma_1|\Psi\rangle_B \\ & +\tfrac{1}{2}|\Phi_2\rangle_{AC}\otimes\sigma_2|\Psi\rangle_B+\tfrac{1}{2}|\Phi_3\rangle_{AC}\otimes\sigma_3|\Psi\rangle_B\,. \end{aligned} \tag{4.74}$$

The above right-hand side allows us to understand what happens when Alice measures the state of (A,C) in the Bell basis. If she measures:

- $|\Phi_0\rangle_{AC}$, then the B state becomes $|\Psi\rangle_B$,
- $|\Phi_1\rangle_{AC}$, then the B state becomes $\sigma_1|\Psi\rangle_B$,
- $|\Phi_2\rangle_{AC}$, then the B state becomes $\sigma_2|\Psi\rangle_B$,
- $|\Phi_3\rangle_{AC}$, then the B state becomes $\sigma_3|\Psi\rangle_B$.

If Alice got $|\Phi_0\rangle_{AC}$ then Bob is in the possession of the teleported state and has to do nothing. If Alice gets $|\Phi_i\rangle_{AC}$, Bob's particle is goes into the state $\sigma_i|\Psi\rangle_B$. Bob applies the i-th box, which multiplies his state by σ_i giving him the desired state $|\Psi\rangle_B$. The teleporting is thus complete!

Note that Alice is left with one of the Bell states $|\Phi_\mu\rangle_{AC}$ which has no information whatsoever about the constants α and β that defined the state to be teleported. Thus the process did not create a copy of the state. The original state is destroyed in the process of teleportation.

It is noteworthy that all the mathematical work above led to the key result (4.74), which is neatly summarized as the following identity valid for arbitrary states $|\Psi\rangle$:

$$\boxed{|\Phi_0\rangle_{AB}\otimes|\Psi\rangle_C \;=\; \frac{1}{2}\sum_{i=0}^{3}|\Phi_i\rangle_{AC}\otimes\sigma_i|\Psi\rangle_B\,.} \tag{4.75}$$

This is an identity for a state of three particles. It expresses the tensor product of an entangled state of the first two particles, times a third, as a sum of products that involve entangled states of the first and third particle times a state of the second particle.

5 EPR and Bell Inequalities

In this section we begin by studying some properties of the singlet state of two particles of spin-1/2. We then turn to the claims of Einstein, Podolsky, and Rosen (EPR) concerning quantum mechanics. Finally, we discuss the so-called Bell inequalities that would follow if EPR were right. Of course, quantum mechanics violates these inequalities, and experiment shows that the inequalities are indeed violated. EPR were wrong.

We have been talking about the singlet state of two spin-1/2 particles. This state emerges, for example, in particle decays. The neutral η_0 meson (of rest mass 547 MeV) sometimes decays into two oppositely charged muons

$$\eta_0 \to \mu^+ + \mu^- \,. \tag{5.1}$$

The meson is a spinless particle and being at rest has zero orbital angular momentum. As a result it has zero total angular momentum. As it decays, the final state of the two muons must have zero total angular momentum as well. If the state of the two muons has zero orbital angular momentum, it must also have zero total spin angular momentum. The two muons flying away from each other with zero orbital angular momentum are in a singlet state. This state takes the form

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\big(|+\rangle_1|-\rangle_2 - |-\rangle_1|+\rangle_2\big)\,. \tag{5.2}$$

This singlet state is rotational invariant and therefore it is actually the same for whatever choice of direction$\mathbf{n}$ to define a basis of spin states:

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|\mathbf{n};+\rangle_1|\mathbf{n};-\rangle_2 - |\mathbf{n};-\rangle_1|\mathbf{n};+\rangle_2\big)\,. \tag{5.3}$$

We now ask: In this singlet, what is the probability $P(\mathbf{a},\mathbf{b})$ that the first particle is in the state $|\mathbf{a};+\rangle$ and the second particle is in the state $|\mathbf{b};+\rangle$, with $\mathbf{a}$ and $\mathbf{b}$ two arbitrarily chosen unit vectors? To help ourselves, we write the singlet state using the first vector

$$|\Psi\rangle \;=\; \frac{1}{\sqrt{2}}\big(|\mathbf{a};+\rangle_1|\mathbf{a};-\rangle_2 - |\mathbf{a};-\rangle_1|\mathbf{a};+\rangle_2\big)\,. \tag{5.4}$$

By definition, the probability we want is

$$P(\mathbf{a},\mathbf{b}) \;=\; \Big|{}_1\langle\mathbf{a};+|_2\langle\mathbf{b};+|\Psi\rangle\Big|^2 \tag{5.5}$$

Only the first term in (5.4) contributes and we get

$$P(\mathbf{a},\mathbf{b}) \;=\; \frac{1}{2}\big|\langle\mathbf{b};+|\mathbf{a};-\rangle\big|^2 \tag{5.6}$$

We recall that the overlap-squared between two spin states is given by the cosine-squared of half the angle in between them. Using figure 2 we see that the angle between $\mathbf{b}$ and $-\mathbf{a}$ is $\pi - \theta_{ab}$, where θ_{ab} is the angle between $\mathbf{b}$ and $\mathbf{a}$. Therefore

$$P(\mathbf{a},\mathbf{b}) \;=\; \tfrac{1}{2}\cos^2\big(\tfrac{1}{2}(\pi-\theta_{ab})\big) \tag{5.7}$$

Our final result is therefore

$$\boxed{P(\mathbf{a},\mathbf{b}) \;=\; \tfrac{1}{2}\sin^2\!\left(\tfrac{1}{2}\theta_{ab}\right).} \tag{5.8}$$

As a simple consistency check, if $\mathbf{b} = -\mathbf{a}$ then $\theta_{ab} = \pi$ and $P(\mathbf{a}, -\mathbf{a}) = 1/2$ which is what we expect.

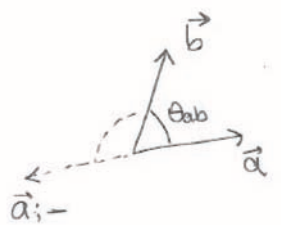

Figure 2: Directions associated with the vectors **a** and **b**.

If we measure about orthogonal vectors, like the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ we get

$$P(\hat{\mathbf{z}}, \hat{\mathbf{x}}) = \tfrac{1}{2}\sin^2 45^\circ = \tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{4}\,. \tag{5.9}$$

The key statement of Einstein, Podolsky and Rosen (EPR) is the claim for **local realism**. This is posed as two properties of measurement:

1. The result of a measurement corresponds to some element of reality. If a measurement of an observable gives a value, that value was a property of the state.

2. The result of a measurement at one point cannot depend on whatever action takes place at a far away point at the same time.

Both properties seem quite plausible at first thought. The first, we are by now accustomed, is violated in Quantum Mechanics, where measurement involves collapse of the wavefunction, so that the result was not pre-ordained and does not correspond to a unequivocal property of the system. The violation of the second is perhaps equally disturbing, given our intuition that simultaneous spatially separated events can't affect each other. There is something non-local about quantum mechanics.

According to EPR the so called entangled pairs are just pairs of particles that have definite spins. They point out that the results of quantum mechanical measurements are reproduced if our large ensemble of pairs has the following distribution of states:

- In 50% of pairs, particle 1 has spin along $\hat{\mathbf{z}}$ and particle 2 has spin along $-\hat{\mathbf{z}}$,

- In 50% of pairs, particle 1 has spin along $-\hat{\mathbf{z}}$ and particle 2 has spin along $\hat{\mathbf{z}}$.

This would explain the perfect correlations and is consistent, for example, with $P(\hat{\mathbf{z}}, -\hat{\mathbf{z}}) = 1/2$, which we obtained quantum mechanically.

The challenge for EPR is to keep reproducing the results of more complicated measurements. Suppose each of the two observers can measure spin along two possible axes: the x and z axes. They measure once, in any of these two directions. EPR then state that in any pair each particle has a definite state of spin in these two directions. For example, a particle of type $(\hat{\mathbf{z}}, -\hat{\mathbf{x}})$ is one that if measured along z gives a plus $\hbar/2$ and if measured along x gives $-\hbar/2$. We do not do simultaneous measurements or subsequent measurements on each particle. EPR then claim that the observed quantum mechanical results are matched if our ensemble of pairs have the following properties

- 25% of pairs have particle 1 in $(\hat{\mathbf{z}}, \hat{\mathbf{x}})$ and particle 2 in $(-\hat{\mathbf{z}}, -\hat{\mathbf{x}})$
- 25% of pairs have particle 1 in $(\hat{\mathbf{z}}, -\hat{\mathbf{x}})$ and particle 2 in $(-\hat{\mathbf{z}}, \hat{\mathbf{x}})$
- 25% of pairs have particle 1 in $(-\hat{\mathbf{z}}, \hat{\mathbf{x}})$ and particle 2 in $(\hat{\mathbf{z}}, -\hat{\mathbf{x}})$
- 25% of pairs have particle 1 in $(-\hat{\mathbf{z}}, -\hat{\mathbf{x}})$ and particle 2 in $(\hat{\mathbf{z}}, \hat{\mathbf{x}})$

First note the complete correlations: particles one and two have opposite spins in each possible direction. This is, of course, needed to match the quantum mechanical singlets. We can ask what is $P(\hat{\mathbf{z}}, -\hat{\mathbf{z}})$, the probability that particle one is along $\hat{\mathbf{z}}$ and particle two along $-\hat{\mathbf{z}}$. The first two cases above apply, and thus this probability is $1/2$, consistent with quantum mechanics. We can also ask for $P(\hat{\mathbf{z}}, \hat{\mathbf{x}})$. This time only the second case applies giving us a probability of $1/4$ as we obtained earlier in (5.9). The quantum mechanical answers indeed arise.

The insight of Bell was that with Stern-Gerlach apparatuses that could measure in three directions one gets in trouble. Suppose each observer can measure along any one of the three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Again, each particle is just measured once. Let us assume that we have a large number N of pairs that, following EPR, contain particles with well-defined spins on these three directions. A particle of type $(\mathbf{a}, -\mathbf{b}, \mathbf{c})$, for example, if measured along $\mathbf{a}$ would give $\hbar/2$, if measured along $\mathbf{b}$ would give $-\hbar/2$ and if measured along $\mathbf{c}$ would give $\hbar/2$. The following distribution is given:

Populations	**Particle 1**	**Particle 2**
N_1	$(\ \mathbf{a},\ \mathbf{b},\ \mathbf{c})$	$(-\mathbf{a}, -\mathbf{b}, -\mathbf{c})$
N_2	$(\ \mathbf{a},\ \mathbf{b}, -\mathbf{c})$	$(-\mathbf{a}, -\mathbf{b},\ \mathbf{c})$
N_3	$(\ \mathbf{a}, -\mathbf{b},\ \mathbf{c})$	$(-\mathbf{a},\ \mathbf{b}, -\mathbf{c})$
N_4	$(\ \mathbf{a}, -\mathbf{b}, -\mathbf{c})$	$(-\mathbf{a},\ \mathbf{b},\ \mathbf{c})$
N_5	$(-\mathbf{a},\ \mathbf{b},\ \mathbf{c})$	$(\ \mathbf{a}, -\mathbf{b}, -\mathbf{c})$
N_6	$(-\mathbf{a},\ \mathbf{b}, -\mathbf{c})$	$(\ \mathbf{a}, -\mathbf{b},\ \mathbf{c})$
N_7	$(-\mathbf{a}, -\mathbf{b},\ \mathbf{c})$	$(\ \mathbf{a},\ \mathbf{b}, -\mathbf{c})$
N_8	$(-\mathbf{a}, -\mathbf{b}, -\mathbf{c})$	$(\ \mathbf{a},\ \mathbf{b},\ \mathbf{c})$

As required, all spins are correlated in particles one and two. We also have $N = \sum_{i=1}^{8} N_i$. We now record the following probabilities that follow by inspection of the table:

$$P(\mathbf{a},\mathbf{b}) = \frac{N_3 + N_4}{N}\,, \quad P(\mathbf{a},\mathbf{c}) = \frac{N_2 + N_4}{N}\,, \quad P(\mathbf{c},\mathbf{b}) = \frac{N_3 + N_7}{N}\,. \tag{5.10}$$

Consider now the trivially correct inequality:

$$N_3 + N_4 \;\leq\; N_3 + N_7 \;+\; N_2 + N_4\,, \tag{5.11}$$

that on account of (5.10) implies the **Bell inequality**

$$\boxed{P(\mathbf{a},\mathbf{b}) \;\leq\; P(\mathbf{a},\mathbf{c}) + P(\mathbf{c},\mathbf{b})\,.} \tag{5.12}$$

If true quantum mechanically, given (5.8) we would have

$$\tfrac{1}{2}\sin^2\tfrac{1}{2}\theta_{ab} \;\leq\; \tfrac{1}{2}\sin^2\tfrac{1}{2}\theta_{ac} \;+\tfrac{1}{2}\sin^2\tfrac{1}{2}\theta_{cb}\,. \tag{5.13}$$

But this is violated for many choices of angles. Take, for example, the planar configuration in Fig. 3:

$$\theta_{ab} = 2\theta\,, \quad \theta_{ac} = \theta_{cb} = \theta\,. \tag{5.14}$$

For this situation, the inequality becomes

$$\tfrac{1}{2}\sin^2\theta \leq \sin^2\tfrac{1}{2}\theta\,. \tag{5.15}$$

This fails for sufficiently small θ: $\frac{1}{2}\theta^2 \leq \frac{\theta^2}{4}$ is just plain wrong. In fact, the inequality goes wrong for any $\theta < \frac{\pi}{2}$. Experimental results have confirmed that Bell inequalities are violated and thus the original claim of local realism by EPR is wrong.

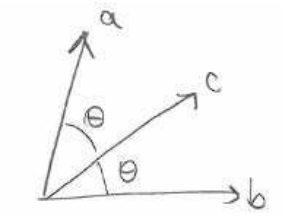

Figure 3: Special configuration for vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

8.05 Quantum Physics II
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