# 15.401 Recitation

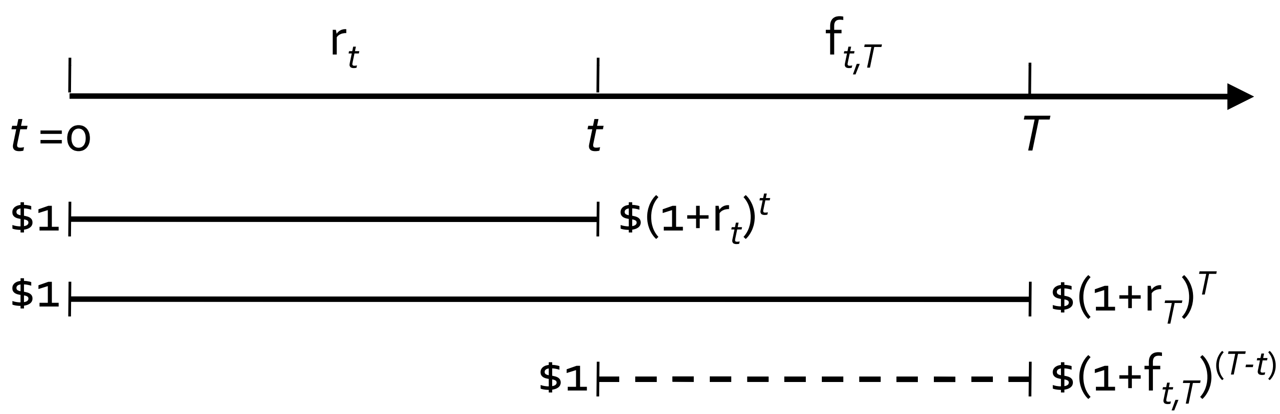
## 2a: Fixed-Income Securities

# Learning Objectives

- Review of Concepts
  - Spot/forward interest rates
  - YTM and bond pricing
- Examples
  - Spot/forward
  - YTM and price
  - Rate of return

# Review: spot/forward interest rates

- **Spot rate** ($r_t$) is the interest rate for the period (0, $t$).
- **Forward rate** ($f_{t,T}$) is the interest rate for the period ($t$, $T$) determined at time 0.
- No arbitrage implies $(1+r_t)^t \times (1+f_{t,T})^{(T-t)} = (1+r_T)^T$.



# Review: zero-coupon bond

- The spot rates are implied in the prices of zero-coupon (pure discount) bonds.
- We can calculate $r_t$ given the price of a *t*-period zero-coupon bond:

$$P = \frac{FV}{(1+r_t)^t} \Leftrightarrow r_t = \left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1$$

(FV = face value)

- After we find $r_t$ and $r_T$, we can calculate $f_{t,T}$.

# Review: coupon bond

- The price of a coupon bond can be expressed as:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+y)^t} + \frac{FV}{(1+y)^T} \quad \text{or} \quad P = \sum_{t=1}^{T} \frac{C_t}{(1+r_t)^t} + \frac{FV}{(1+r_T)^T}$$

- *y* is the yield-to-maturity (or yield). It is equal to the rate of return on the bond if
  - it is bought now at price *P* and held to maturity, and
  - all coupons are reinvested at rate *y*.
- *y* is not a spot rate.
- There is a one-to-one mapping between *y* and *P*.

# Example 1: spot and forward rates

- Yields on three Treasury notes are given as follows:

| Maturity (yrs) | Coupon rate (%) | YTM (%) |
|---|---|---|
| 1 | 0 | 5.25 |
| 2 | 5 | 5.50 |
| 3 | 6 | 6.00 |

a. What are the prices of the 1-year, 2-year and 3-year notes with face value = $100?
b. What are the spot interest rates for year 1, 2 and 3?
c. What is the implied forward rate for year 2 to year 3?

# Example 1: spot and forward rates

- Answer:

a.

$$P_1 = \frac{100}{1+5.25\%} = \$95.01$$

$$P_2 = \frac{5}{1+5.50\%} + \frac{105}{(1+5.50\%)^2} = \$99.08$$

$$P_3 = \$100$$

b. $r_1 = 5.2500\%;\ r_2 = 5.5063\%;\ r_3 = 6.0359\%$

c. $$f_{2,3} = \frac{(1+r_3)^3}{(1+r_2)^2} - 1 = 7.1032\%$$

# Example 2: YTM and price

- What is the price of a ten-year 5% treasury bond (face value = $100, annual coupon payments) if the yield to maturity is…
  - 4%?
  - 5%?
  - 6%?
- When is the price above/at/below par?

# Example 2: YTM and price

- Answer:

$$P(FV, r, y) = \sum_{t=1}^{T} \frac{FV \cdot r}{(1+y)^t} + \frac{FV}{(1+y)^T}$$

$$= FV\left[\frac{r}{y}\left(1 - \frac{1}{(1+y)^T}\right) + \frac{1}{(1+y)^T}\right].$$

  - 4%: \$108.11
  - 5%: \$100.00
  - 6%: \$ 92.64
- Price is above/at/below par when YTM is lower than/equal to/higher than the coupon rate.

# Example 3: Rate of Return

- Suppose that you bought a 2-year STRIP (face value = $100) a year ago, and the interest rates at the time were as follows:

| Years | Spot rate (%) |
|---|---|
| 1 | 2.5 |
| 2 | 3 |
| 3 | 5 |

  - You sell your STRIP right now, and the yield curve happens to be the same as a year ago. What is the annualized return on your investment?
  - What is the annualized return if you sell it next year?

# Example 3: Rate of Return

- Answer:
  - Purchase price = $100/(1.03)^2$ =\$94.26
  - Current price = 100/1.025=\$97.56
  - Sell now: realized return = 3.5024% per year
  - Sell next year: return = 3% per year (for sure)

# Example 3: Rate of Return (revisited)

- Suppose that five years ago today, you bought a 6% ten-year treasury bond (face value = $100, annual coupon payments) at a yield of 3.5% per year.
- Since then, you have deposited the coupons in a bank at 2% per year.
- Today you sell the bond at a yield of 5% per year.
- What is the annualized return on your investment?

# Example 3: Rate of Return (revisited)

- Answer:
  - Cumulative value of deposited coupons = 31.22
  - Selling price today = 104.33
  - Total payoff = 135.55
  - Purchase price = 120.79
  - Annualized realized return = 2.3328%
- Follow-up question:
  - Why is the realized return so low?

# 15.401 Recitation

## 2b: Fixed-Income Securities

# Learning Objectives

- Review of Concepts
  - Bond arbitrage
  - Duration/convexity
  - Immunization
- Examples
  - Duration
  - Bond arbitrage
  - True/false

# Review: bond arbitrage

- Bond arbitrage is possible when its price is not equal to the PV of payments discounted at the spot rates
- Caveats:
  - the bond must have the same risk characteristics as the securities from which the spot rates are derived (e.g., riskless);
  - each coupon payment can be matched exactly by a spot rate;
  - it is possible to borrow/lend at all spot rates.
- General strategy:
  - Buy low, sell high

# Review: bond arbitrage

- Detailed strategy:
  - Scale available payoff streams so that the net cash flow at $t$ = 1, 2, … is exactly zero.
  - Adjust the signs so that the payoff at $t$ = 0 is positive.

| Multiplier | Asset | $t = 0$ | $t = 1$ | $t = 2$ | … | $t = T$ |
|---|---|---|---|---|---|---|
| $m_A$ | A | $-m_A P_A$ | $m_A C_{A1}$ | $m_A C_{A1}$ | … | $m_A C_{AT}$ |
| $m_B$ | B | $-m_B P_B$ | $m_A C_{A1}$ | $m_B C_{B2}$ | … | $m_B C_{BT}$ |
| … | … | … | … | … | … | … |
| $m_N$ | N | $-m_N P_N$ | $m_N C_{N1}$ | $m_N C_{N2}$ | … | $m_N C_{NT}$ |
| + | | $\pi_0$ | 0 | 0 | … | 0 |

# Review: bond arbitrage

- Remarks:
  - Arbitrage strategy is not unique.
  - Given an arbitrage strategy $(m_A, m_B, \ldots, m_N)$ with profit $\pi_0$, $(k\cdot m_A, k\cdot m_B, \ldots, k\cdot m_N)$ is also an arbitrage strategy with profit $k\cdot\pi_0$.
  - A strategy where cash flows at $t = 0, 1, \ldots T$ are all zero except at $t = s > 0$ (when it is positive) is also an arbitrage.
  - For the purpose of this course, we only consider the type of arbitrage strategies on the previous page.

# Review: duration/convexity

- Duration and modified duration measure a bond's exposure to interest rate risk:

$$D = \frac{1}{P}\sum_{t=1}^{T}\frac{t \cdot C_t}{(1+y)^t} + \frac{T \cdot FV}{(1+y)^T}; \quad MD = \frac{D}{1+y}$$

- Since $MD = -\frac{1}{P}\cdot\frac{\partial P}{\partial y}$, a small change ($\Delta y$) in YTM will cause bond price to change by approximately $\Delta P \approx -P \cdot MD \cdot \Delta y$.
- The formula is **not accurate** for large changes in *y*.

# Review: duration/convexity

- Convexity is…
  - the second derivative of P(y);
  - a measure of curvature of P(y);
  - the sensitivity of the duration to a change in the yield.

$$CX = -\frac{1}{P} \cdot \frac{\partial^2 P}{\partial y^2}$$

- A better approximation:

$$\frac{\Delta P}{P} \approx -\Delta y \cdot MD + \frac{(\Delta y)^2}{2} \cdot CX.$$

# Immunization

- The duration of a portfolio with weight $w$ on asset $X$ and (1-$w$) on asset $Y$ is [ $w \times D(X) + (1-w) \times D(Y)$ ].
- Institutions such as banks, pension funds and insurance companies are highly exposed to interest rate fluctuations. They would like to insure or immunize against such fluctuations.
- Solution: structure the balance sheet so that V(Assets) × D(Assets) − V(Liabilities) × D(Liabilities).
- Continuous rebalancing is required for perfect immunization.

# Example 1: duration

- Consider a 10-year bond with a face value of \$100 that pays an annual coupon of 8%. Assume spot rates are flat at 5%.
  a. Find the bond's price and duration.
  b. Suppose that 10yr yields increase by 10bps. Calculate the change in the bond's price using your bond pricing formula and then using the duration approximation.
  c. Suppose now that 10yr yields increase by 200bps. Repeat your calculations for part (b).

# Example 1: duration

- Answer:
  a. $$P = \frac{8}{1.05} + \frac{8}{1.05^2} + \ldots + \frac{108}{1.05^{10}} = \$123.16$$
  $$D = \frac{1}{123.16}\left(\frac{8\cdot 1}{1.05} + \frac{8\cdot 2}{1.05^2} + \cdots + \frac{108\cdot 10}{1.05^{10}}\right) = 7.54$$
  b. Actual new price = \$122.28.
  $$\Delta P \approx -P \times \frac{D}{1+y} \times \Delta y = -123.16 \times \frac{7.54}{1.05} \times 0.001 = -\$0.88$$
  $$\Rightarrow P_{new} = \$122.28.$$
  c. Actual new price = \$107.02
  New price using duration approximation = \$105.47

# Example 2: bond arbitrage

- Find an arbitrage portfolio given the following riskless bonds:

| Asset | $t - 0$ | $t - 1$ | $t - 2$ | $t - 3$ |
|---|---|---|---|---|
| A | -97 | 100 | | |
| B | -92 | | 100 | |
| C | -87 | | | 100 |
| D | -102 | 5 | 5 | 105 |

# Example 2: bond arbitrage

- Answer:

| Multiplier | Asset | *t* = 0 | *t* = 1 | *t* = 2 | *t* = 3 |
|---|---|---|---|---|---|
| x | A | -97x | 100x | | |
| y | B | -92y | | 100y | |
| z | C | -87z | | | 100z |
| w | D | -102w | 5w | 5w | 105w |
| | | | 0 | 0 | 0 |

- x − -0.05w
- y = -0.05w
- z = -1.05w

# Example 2: bond arbitrage

- $\pi_0 = -97 \cdot (-0.05w) - 92 \cdot (-0.05w) - 87 \cdot (-1.05w) - 102w$
  $= -1.2w$
- Set w = -1
- Arbitrage strategy:
  - Long 0.05 A
  - Long 0.05 B
  - Long 1.05 C
  - Short 1 D
- Profit = 1.2

# Example 3: true or false

- True or false:
  - Investors expect higher returns on long-term bonds than short-term bonds because they are riskier. Thus, the term structure of interest rates is always upward sloping.
  - To reduce interest rate risk, an overfunded pension fund, i.e., a fund with more assets than liabilities, should invest in assets with longer duration than its liabilities.

# Example 3: true or false

- Answer:
  - False. The term structure depends on the expected path of interest rates (among other factors). For example, if interest rates are expected to fall, the term structure will be downward sloping.
  - False. To minimize interest rate risks, we want MD(A) x V(A) – MD(L) x V(L) = 0. If V(A) > V(L), we want MD(A) < MD(L). That means we should invest in assets with shorter duration.

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