18.01 Practice Exam 3 Sol'ns Fall 2006

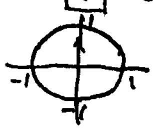
$\boxed{1}$ a) $\cos 2x = \cos^2 x - \sin^2 x$
$= (1-\sin^2 x) - \sin^2 x$
$= 1 - 2\sin^2 x$

$$\int \sin^2 x\, dx = \int \frac{(1-\cos 2x)}{2}\, dx$$
$$= \frac{x}{2} - \frac{\sin 2x}{4} + c$$

b) $D(x \ln x) = \ln x + x \cdot \frac{1}{x}$
$= \ln x + 1$

$\therefore$ by the fundamental theorem,
$$x \ln x \Big]_1^e = \int_1^e \ln x\, dx + \int_1^e 1\, dx$$
$$e \cdot 1 - 0 = \int_1^e \ln x\, dx + e - 1$$
$$\therefore \int_1^e \ln x\, dx = 1.$$

$\boxed{2}$ By horizontal slices,
(calculate vol. of top half & double it)



$$= \int_0^1 \pi x^2\, dy = \pi \int_0^1 (1-y^4)\, dy$$
$$= \pi \left(y - \frac{y^5}{5}\right)\Big|_0^1 = \pi \cdot \frac{4}{5}$$

By cylindrical shells: $y = (1-x^2)^{1/4}$
$$= \int_0^1 2\pi x \cdot (1-x^2)^{1/4}$$
$$= -\frac{4\pi}{5}(1-x^2)^{5/4}\Big]_0^1 = \frac{4\pi}{5}$$

$\therefore$ Volume is $\frac{8\pi}{5} \approx \frac{8 \cdot (3.14)}{5} > \frac{25}{5}$

5 cubic feet is not enough.

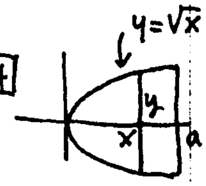
$\boxed{3}$ a) $F(x) = \int_0^x t^2 e^{-t^2}\, dt$; $F'(x) = x^2 e^{-x^2}$
(second fund thm)

b) $F' = 0$ when $x = 0$; otherwise $F'(x) > 0$. Thus F is increasing, so $x = 0$ is a point of horiz. inflection (not a max or min)

c) $u = t^2$; $du = 2t\,dt$ $\int_0^9 \sqrt{u}\, e^{-u}\, du = \int_0^3 t e^{-t^2} \cdot 2t\, dt$
$= 2 \cdot F(3)$

d) $e^{-t^2} \leq 1$
$$\therefore \int_0^x t^2 e^{-t^2}\, dt \leq \int_0^x t^2\, dt = \frac{x^3}{3}.$$

$\boxed{4}$



Area of slice at x is $\pi y^2 = \pi x$

Average area of slices $= \frac{1}{a}\int_0^a \pi x\, d$
$$= \frac{1}{a}\, \pi \frac{x^2}{2}\Big]_0^a$$

Therefore
average area $= \frac{\pi a}{2}$
which is the area of the slice at $x_0 = a/2$ (halfway)
$$\pi \cdot (\sqrt{x_0})^2 = \frac{\pi a}{2} \Rightarrow x_0 = a/2$$

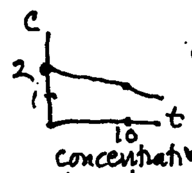
$\boxed{5}$

1	8	15	22	29
3	2	0	1	3

a) by trapezoidal rule:
Total # hits $\approx \left(\frac{3}{2} + 2 + 0 + 1 + \frac{3}{2}\right) 7 = 6 \cdot 7$
$= \boxed{42}$

b) by Simpson's rule:
Total # hits $\approx \left(\frac{3 + 4\cdot 2 + 2\cdot 0 + 4 \cdot 1 + 3}{6}\right) \cdot 14$
$= \frac{18}{6} \cdot 14 = \boxed{42}$

$\boxed{6}$ In an infinitesimal time interval dt at time t,



concentration over time
slope $= -\frac{1}{10}$

$C = 2 - \frac{1}{10} t$

flow rate of water at time t into pool (cc/hour) $= t^2 (10-t)^2 \cdot 10^4$

pool surface is $(100\text{ cm})^2$

$\therefore$ amt entering from time t to $t + dt$
$\circledast \quad = t^2 (10-t)^2 \cdot 10^4 \cdot \left(2 - \frac{t}{10}\right) \cdot d$

Total amt $\doteq 10^4 \int_0^{10} t^2 (10-t)^2 \cdot \left(2 - \frac{t}{10}\right) dt$ $\circledast\circledast$
nanograms

For Δt calculation:
replace dt by Δt in $\circledast$
write $\circledast\circledast$ as $\sum_1^n 10^4 t_i^2 (10-t_i)^2 (2 - t_i/10) \Delta$
and pass to limit as $n \to \infty$ $\rightarrow$ integral given.