18.01 Practice Exam Solutions

1. a) $$\frac{d}{dt}\left(\frac{3t}{\ln t}\right)\Big|_{e^2} = \frac{3\ln t - 3t\cdot\frac{1}{t}}{(\ln t)^2}\Big|_{t=e^2}$$

$$= \frac{3\ln(e^2)-3}{(\ln(e^2))^2} = \boxed{\frac{3}{4}}$$

b) $$\lim_{u\to 0}\frac{3u}{\tan(2u)} = \lim_{u\to 0}\frac{3u}{\frac{\sin 2u}{\cos 2u}} = \lim_{u\to 0}\frac{3u\cdot\cos(2u)}{\sin(2u)}$$

$$=\left(\lim_{u\to 0} 3\cos(2u)\right)\left(\frac{1}{2}\lim_{u\to 0}\frac{2u}{\sin(2u)}\right)$$

$$= 3\cdot\left(\frac{1}{2}\cdot 1\right) = \boxed{3/2}.$$ The 3^{rd} equality holds because the ~~[illegible]~~ limit of the product is the product of the limits, when both limits exist (and one is not equal to $\pm\infty$ while the other is 0). The fourth equality holds because limit $\lim_{u\to 0}\frac{u}{\sin u} = 1$.

c) $$\frac{d^3}{dx^3}(\sin(kx)) = \frac{d^2}{dx^2}(k\cos(kx)) = \frac{d}{dx}(-k^2\sin(kx))$$

$$= \boxed{-k^3\cos(kx).}$$

d) $$\frac{d}{d\theta}\left[(a+k\sin^2\theta)^{1/3}\right] = \frac{1}{3}(a+k\sin^2\theta)^{-2/3}\cdot\frac{d}{d\theta}(k\sin^2\theta)$$

$$= \boxed{\frac{1}{3\sqrt[3]{(a+k\sin^2\theta)^2}}\cdot 2k\sin\theta\cos\theta}.$$

2. $$\left.\frac{d}{dx}(x^3)\right|_{x_0} = \lim_{\Delta x \to 0} \frac{(x_0+\Delta x)^3 - x_0^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x_0^3 + 3x_0^2\Delta x + 3x_0(\Delta x)^2 + (\Delta x)^3 - x_0^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x_0^2 + 3x_0(\Delta x) + (\Delta x)^2\right) = \boxed{3x_0^2}.$$

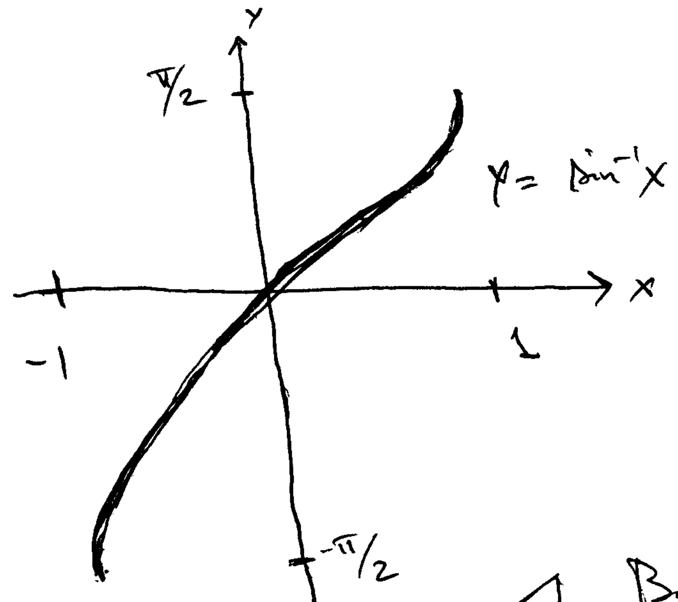
3. Let $f(x) = \sqrt[3]{x}$. Then $f'(1) = \lim_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{\sqrt[3]{1+h} - \sqrt[3]{1}}{h} = \lim_{h \to 0} \frac{\sqrt[3]{1+h} - 1}{h}$$

so $\lim_{h \to 0} \dfrac{1 - \sqrt[3]{1+h}}{h} = -f'(1)$. Now, $f'(x) = \frac{1}{3}x^{-2/3}$,

so $-f'(1) = -\frac{1}{3}\dfrac{1}{\sqrt[3]{1^2}} = \boxed{-1/3}$.

4.

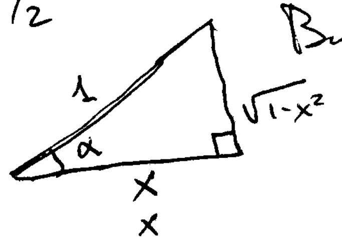


We have $\sin y = \sin \sin^{-1} x = x$

so $\dfrac{d}{dx}(\sin y) = 1$.

$\cos y \dfrac{dy}{dx} = 1$.

so $\dfrac{dy}{dx} = 1/\cos y$



But $\cos y = \sqrt{1-x^2}$, we chose the positive square root because $\sin^{-1}x$ has range $[-\pi/2, \pi/2]$, and cosine is positive on this interval.

So $\boxed{\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}}$.

5. a. $f(x)$ only has a possible discontinuity at $x=0$.

For $f(x)$ to be continuous at $x=0$, we need

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x).$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ax+b) = b$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (1-x+x^2) = 1.$$ So for f to be continuous at $x=0$, b must equal 1, and a can be arbitrary.

b. $$f'(x) = \begin{cases} a & x>0 \\ -1+2x & x<0 \end{cases}$$

$$\lim_{x\to 0^+} f'(x) = a \qquad \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (-1+2x) = -1$$

So a must equal -1 for f to be differentiable.

Actually, we must go a step further. <u>Neither</u> formula yields $f'(0)$. We must make sure $f'(0)$ exists separately.

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h}.$$ We need $\lim_{h\to 0^+} (\cdots)$ to exist

Note h can be positive or negative!

and $\lim_{h\to 0^-} (\cdots)$ needs to exist

and they must be equal.

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{ah+b-1}{h} = a = -1$$

$$\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{1-h+h^2-1}{h} = \lim_{h\to 0^-}(-1+h) = -1.$$

So $a=-1$, $b=1$, makes $f(x)$ continuous and differentiable (and the derivative continuous).

6. The tangent line is horizontal when $\frac{dy}{dx}=0$.
We differentiate implicitly and use the product rule:

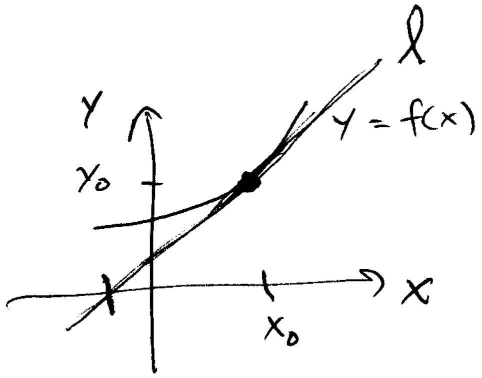
$$2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx} + 2x = 0$$

$$\frac{dy}{dx}(x^2+3y^2) = -2x(1+y).$$

Hence, if $\frac{dy}{dx}=0$, then $x=0$ or $y=-1$.

If $x=0$, then $y^3=8$, so $y=2$.

If $y=-1$, then $-x^2-1+x^2=8$, which is impossible. So the only point where the tangent line to $x^2y+y^3+x^2=8$ is horizontal is $(0,2)$.

7. 

Let ℓ be the tangent line to the graph $y=f(x)$ at (x_0,y_0). Then the equation of ℓ is:

$y_0 = f(x_0)$

$$y-y_0 = f'(x_0)(x-x_0).$$

At the x-intercept $y=0$. So we get

$$-y_0 = f'(x_0)(x-x_0), \text{ or using } y_0=f(x_0),$$

$$-f(x_0) = f'(x_0)(x-x_0). \text{ If } f'(x_0)\neq 0,$$

then $x-x_0 = -\dfrac{f(x_0)}{f'(x_0)}$, so

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$ If $f'(x_0)=0$, then the tangent line ℓ is parallel to the x-axis and never intersects it, unless $y_0=0$, in which case ℓ coincides with the x-axis.

8] $V = \frac{4}{3}\pi r^3$. We are given that

$$\left.\frac{dV}{dt}\right|_{r=20\,cm} = -10\ cm^3/s.$$ Differentiating

the formula for volume we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}. \quad \text{So} \quad \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}.$$

So $$\left.\frac{dr}{dt}\right|_{r=20\,cm} = \frac{1}{4\pi(20\,cm)^2}(-10)\ cm^3/s$$

$$= \boxed{\frac{-1}{160\pi}\ cm/s.}$$

9] a) $\sec x = \frac{1}{\cos x}$. The discontinuities are at points where $\cos x = 0$, i.e. $x = \frac{\pi}{2} + k\pi$, k an integer.

b) $\frac{1+x^2}{1-x^2}$ has discontinuities where the denominator is 0, i.e. $1 - x^2 = 0$, so $\boxed{x = \pm 1}$.

c) $$\frac{d}{dx}|x| = \begin{cases} +1 & x > 0 \\ -1 & x < 0 \end{cases}$$ and undefined at $x = 0$.

So there is a jump-discontinuity at $x = 0$.

10) a) $A = A_0 e^{-rt}$. Suppose $A(t) = \frac{1}{4} A_0$.

Then we get $\frac{1}{4} A_0 = A_0 e^{-rt}$, or $\frac{1}{4} = e^{-rt}$ (since $A_0 > 0$). $-\ln 4 = -rt$, or

$\boxed{t = \frac{\ln 4}{r}}$. So it takes $\frac{\ln 4}{r}$ units of time for the amount of material to fall to $\frac{1}{4}$ the original. Note that this is $2\frac{\ln 2}{r}$. $\frac{\ln 2}{r}$ is the amount of time it takes the amount of material to fall to $\frac{1}{2}$ the amount, i.e. the "half-life." The time it takes the ~~quantity~~ of material to fall to $\frac{1}{4}$ the original, is <u>two</u> half-lives.

b)
$$\left.\frac{dA}{dt}\right|_{t=\frac{\ln 4}{r}} = \left.\frac{d}{dt}\left(A_0 e^{-rt}\right)\right|_{t=\frac{\ln 4}{r}}$$

$$= \left.-A_0 r e^{-rt}\right|_{t=\frac{\ln 4}{r}} = -A_0 r e^{-\ln 4}$$

$$= \boxed{-\frac{A_0 r}{4} \text{ grams/sec}}$$