

# 18.01 Solutions for Practice Questions (Exam 1)

[1] a) By quotient rule:

$$\frac{d}{dx} \left( \frac{\sqrt{x}}{1+2x} \right) = \frac{\frac{1}{2\sqrt{x}}(1+2x) - \sqrt{x} \cdot 2}{(1+2x)^2}$$

value at  $x=1$ :  $\frac{\frac{1}{2} \cdot 3 - 1 \cdot 2}{3^2} = -\frac{1}{18}$

b) By product rule:

$$\begin{aligned} \frac{d}{du} (u \ln 2u) &= 1 \cdot \ln 2u + u \cdot \frac{1}{2u} \cdot 2 \\ &= \ln 2u + 1 \end{aligned}$$

[2] a) By two uses of the chain rule:

$$\frac{d}{dt} (1 - k \cos^2 t)^{1/2} = \frac{1}{2} (1 - k \cos^2 t)^{-1/2} \cdot (-2k \cos t) \cdot (-\sin t)$$

$$= \frac{k \cos t \sin t}{\sqrt{1 - k \cos^2 t}} \quad \text{OR} \quad \frac{k \sin 2t}{2 \sqrt{1 - k \cos^2 t}}$$

[b] If  $k=1$ ,  $\sqrt{1 - \cos^2 t} = \sin t$ , so the above becomes  $\cos t$  which agrees with

$$\frac{d}{dt} \sqrt{1 - \cos^2 t} = \frac{d}{dt} \sin t = \cos t$$

$$\begin{aligned} [3] \quad \frac{d}{dx} \left( \frac{1}{x^2} \right) &= \lim_{\Delta x \rightarrow 0} \frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left( \frac{x^2 - (x^2 + 2x\Delta x + (\Delta x)^2)}{(x+\Delta x)^2 \cdot x^2} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{-2x\Delta x}{(x+\Delta x)^2 \cdot x^2} = \frac{-2x}{x^4} \\ &= \boxed{-\frac{2}{x^3}} \end{aligned}$$

[4]  $y = \sin^{-1} x \Rightarrow x = \sin y$

Differentiating:  $1 = \cos y \cdot \frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

(Since  $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ ,  $\cos y \geq 0$  so use positive  $\sqrt{\quad}$ )

[5] Since  $f(x)$  is differentiable, it is also continuous. Thus the two functions must have same value at  $x=1$  (in the limit), and the same slope (in the limit):

$$\begin{aligned} \text{value: } ax+b &= x^2-3x+2 \\ \text{slope: } a &= 2x-3 \Big|_{x=1} = -1 \end{aligned}$$

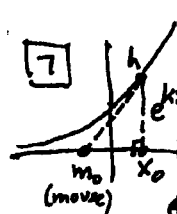
$$\therefore \boxed{a = -1, b = 1}$$

$$[6] \quad a) \lim_{u \rightarrow 0} \frac{\tan 2u}{u} = \lim_{u \rightarrow 0} \frac{\sin 2u}{2u} \cdot \frac{2}{\cos 2u} = 1 \cdot \frac{2}{1} = 2$$

(another way:  $\lim_{u \rightarrow 0} \frac{\sin 2u}{u \cdot \cos 2u} = \lim_{u \rightarrow 0} \frac{\sin u}{u} \cdot \frac{\cos u}{\cos 2u} = 2 \cdot 1 \cdot \frac{1}{1} = 2$ )

b)  $\frac{d}{dx} e^x \Big|_{x=0} = \lim_{h \rightarrow 0} \frac{e^{0+h} - e^0}{h} = \lim_{h \rightarrow 0} \frac{e^h - 1}{h}$   
(can use  $\Delta x$  instead of  $h$ )  
But  $\frac{d}{dx} e^x \Big|_{x=0} = e^x \Big|_{x=0} = 1$

Therefore, the limit is 1.

[7]  Simplest: calculate slope of tangent line two ways:  
 $\frac{e^{kx_0} - m_0}{x_0 - m_0} = \frac{d}{dx} e^{kx} \Big|_{x_0} = k e^{kx_0}$   
 $\therefore x_0 - m_0 = \frac{1}{k}; \quad \boxed{m_0 = x_0 - \frac{1}{k}}$

[OR] Equation of tan. line is  $y - y_0 = \underset{\substack{\text{slope} \\ = \text{deriv.}}}{k e^{kx_0}} (x - x_0); \quad y_0 = e^{kx_0}$

The  $x$ -intercept  $m_0$  is where  $y=0$   
 $\therefore -e^{kx_0} = k e^{kx_0} (m_0 - x_0) \quad (x = m_0 \text{ there})$   
so  $-\frac{1}{k} = m_0 - x_0, \quad \boxed{m_0 = x_0 - \frac{1}{k}}$