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# SOLUTIONS TO CHAPTER 15

## 15.1 SOURCE AND MATERIAL CONFIGURATIONS

### 15.1.1

TABLE P15.1.1. Modal Field Representation

| | Physical Constraints | Example/Prob. |
|---|---|---|
| Cartesian | | |
| Laplace's Eq. | EQS Potential | Sec. 5.5, Demo. 5.5.1 |
| | | Probs. 5.5.1-7 |
| | EQS 3-Dimensional | Examp. 5.10.1 |
| | | Probs. 5.10.1,3 |
| | Polarization | Examp. 6.6.3, 6.7.1 |
| | | Prob. 6.3.10, 6.6.9 |
| | | Prob. 6.7.1 |
| | Conduction | Examp. 7.4.1 |
| | Charge Relax. | Prob. 7.9.12 |
| | MQS, Equi-A | Examp. 8.6.3 |
| | | Demo. 8.6.2 |
| | | Prob. 8.6.10 |
| | Magnetization | Prob. 9.6.9 |
| | MQS E | Prob. 10.1.2 |
| | MQS Eddy Current | Prob. 10.1.5 |
| Poisson's Eq. | EQS Potential | Probs. 5.6.7-9, 13 |
| | MQS Equa-A | Prob. 8.6.7 |
| Polar | | |
| Laplace's Eq. | EQS Potential | Examp. 5.8.2-3 |
| | | Probs. 5.8.3-9 |
| | Conduction | Prob. 7.4.4, 7.5.6 |
| | MQS, Constrained Current | Prob. 8.5.2 |
| | MQS, Equi-A | Prob. 8.6.5 |
| | MQS E | Examp. 10.12 |
| | | Prob. 10.1.3 |
| Initial Value | Diffusion Eq. | Examp. 10.6 |
| | | Prob. 10.6.1-2 |
| Helmholtz Eq. | TM Modes | Examp. 13.3.1 |
| | | Prob. 13.3.1-6 |
| | 3-Dimensional | Probs. 13.4.3-4 |
| | | Demo. 13.3.1 |
| | TE Modes | Examp. 13.3.2 |
| | | Demo. 13.3.2 |

## 15.2 MACROSCOPIC MEDIA

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**15.2.1** In each case, the excitation is an imposed uniform field at infinity. For (a), the field is tangential to the spherical surface everywhere except at the singular points at the poles. Thus, i) the system could be EQS with the regions insulating dielectrics and $\epsilon_a \gg \epsilon_b$, ii) the system could be a stationary conductor with the field lines either **J** or **E** and $\sigma_a \gg \sigma_b$, iii) it could be MQS with the lines **B** or **H**, the materials insulating and $\mu_a \gg \mu_b$ and iv) it could be a perfectly conducting sphere in an insulating media with the lines either **B** or **H** changing in time rapidly enough to induce the currents in the sphere required to exclude the field.

For (b), the field is perpendicular to the surface. Thus, i) it could be EQS and a perfect conductor in an insulating medium with the lines representing **E**, ii) it could be EQS **E** with the materials perfect insulators (the field changing rapidly compared to the charge relaxation time in either material) with $\epsilon_b \gg \epsilon_a$, iii) it could be **J** or **E** in stationary conduction with $\sigma_b \gg \sigma_a$, iv) and it could be MQS **H** or **B** with the materials insulating and $\mu_b \gg \mu_a$.

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**15.2.2** The excitation is inside the sphere. In (a), the field in that region is perpendicular to the interface. Thus, i) the lines could be EQS **E** with the inside an insulator and the outside a perfect conductor, ii) the system could again be EQS and the lines could be **E** with both materials perfect insulators and $\epsilon_a \gg \epsilon_b$, iii) it could be stationary conduction with the lines either **E** or **J** and a dipole current source with $\sigma_a \gg \sigma_b$ and iv) the lines could be MQS **H** or **B** with a magnetic dipole and the regions magnetizable insulators with $\mu_a \gg \mu_b$.

In (b), the interior field lines are tangential to the surface. Thus, i) the dipole could be electric and the materials perfect insulators having $\epsilon_b \gg \epsilon_a$, ii) the dipole could be a current source for stationary conduction with the lines **E** or **J** and $\sigma_b \gg \sigma_a$, iii) the system could be MQS with the dipole magnetic and the materials magnetizable insulators having $\mu_b \gg \mu_a$, and iv) the system could be MQS with a magnetic dipole varying rapidly enough with time to make the outer material a perfect conductor while the interior one remains a perfect insulator.

## 15.3 CHARACTERIC TIMES, PHYSICAL PROCESSES, AND APPROXIMATIONS

**15.3.1** Because it does not involve $\sigma, \omega$ is normalized to $\tau_{em}$. Thus, the horizontal axis is

$$\log(\omega\tau_{em}) = \log(\omega l\sqrt{\mu\epsilon})$$

Then

$$\omega\tau_e = \frac{\omega\epsilon}{\sigma} = \omega\tau_{em}\frac{\epsilon}{\sigma l\sqrt{\mu\epsilon}} = 1 \Rightarrow \omega\tau_{em} = \frac{\sigma}{(\sqrt{\epsilon/\mu}/l)}$$

Thus, with the *characteristic conductivity* defined as

$$\sigma^* \equiv \sqrt{\epsilon/\mu}/l$$

the critical line indicating charge relaxation, $\omega\tau_e = 1$, is written in terms of the independent variables of normalized frequency and conductivity as

$$\log \omega\tau_{em} = \log\left(\frac{\sigma}{\sigma^*}\right)$$

Similarly,

$$\omega\tau_m = 1 \Rightarrow \omega\tau_{em} = \frac{\tau_{em}}{\tau_m} = \frac{l\sqrt{\mu\epsilon}}{\mu\sigma l^2} = \left(\frac{\sigma}{\sigma^*}\right)^{-1} \Rightarrow \log \omega\tau_{em} = -\log\left(\frac{\sigma}{\sigma^*}\right)$$

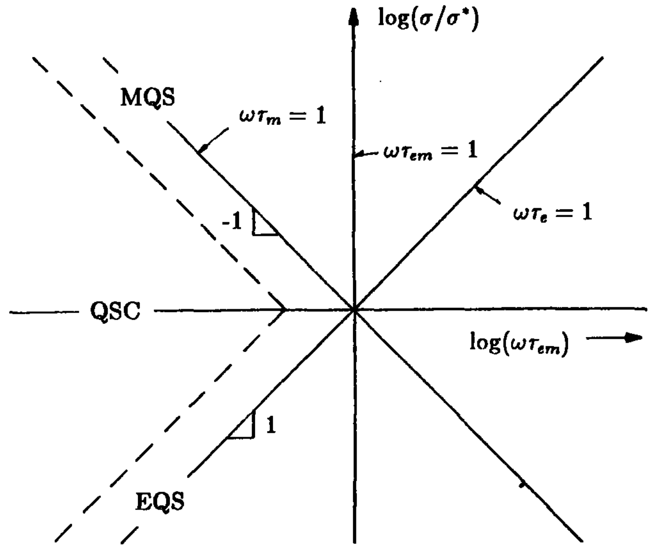



**Figure S15.3.1**

Thus, the plot is as shown in Fig. S15.3.1. If $\sigma > \sigma^*$, raising the frequency results in a transition from stationary conduction to the MQS regime while if $\sigma < \sigma^*$, the transition is to the EQS regime.

**15.3.2** (a) In the limit of zero frequency, the electric and magnetic fields are as summarized by (7.5.7) and (7.5.11) and by (11.3.10) and (11.2.12). With (a) and (b) respectively designating the nonconducting annulus and the rod,

$$\mathbf{E}^b = \frac{v}{L}\mathbf{i}_z \tag{1}$$

$$\mathbf{E}^a = -\frac{v}{ln(a/b)}\left[\frac{z}{rL}\mathbf{i}_r + \frac{ln(r/a)}{L}\mathbf{i}_z\right] \tag{2}$$

$$\mathbf{H}^b = \frac{\sigma v}{L}\frac{r}{2}\mathbf{i}_\phi \tag{3}$$

$$\mathbf{H}^a = \frac{\sigma v}{L}\frac{b^2}{2r}\mathbf{i}_\phi \tag{4}$$

The magnetic field is induced by the uniform current density

$$\mathbf{J} = \frac{\sigma v}{L}\mathbf{i}_z; \qquad 0 < r < b \tag{5}$$

which is returned as the surface current density $\mathbf{K} = -[\sigma v b^2/2La]$ in the perfectly conducting wall. There is no volume charge density in the interior of the rod. On its surface and on the inner surface of the outer wall, the surface charge densities are

$$\sigma_s(r = a) = \frac{\epsilon_o v}{ln(a/b)}\frac{z}{aL}; \qquad \sigma_s(r = b) = -\frac{\epsilon_o v}{ln(a/b)}\frac{z}{Lb} \tag{6}$$

These fields and sources are sketched in Fig. S15.3.2a.

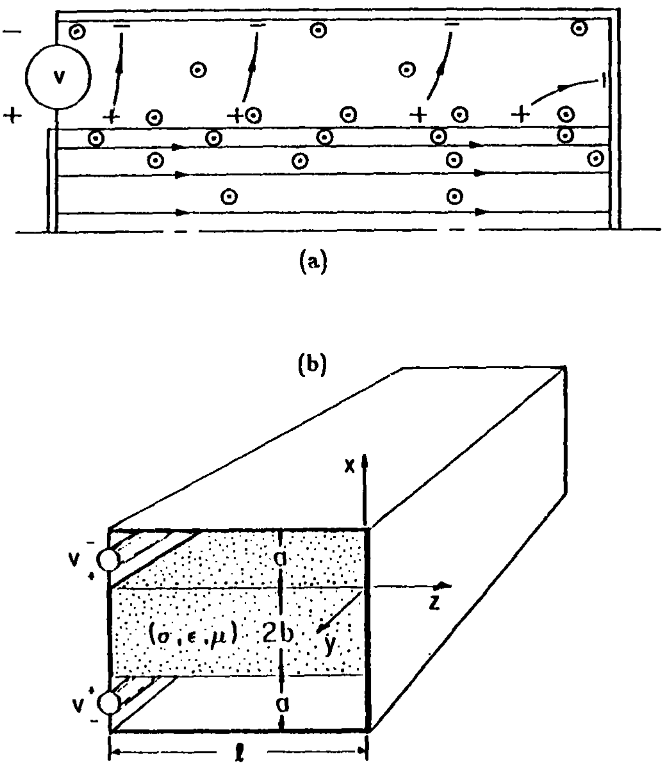



**Figure S15.3.2a,b**

(b) With all dimensions on the same order, the argument is as given in this section. Any one of the dimensions, $a, b$ or $L$ is the typical dimension. The ratio of that dimension to either of the other two is presumed to be perhaps 2 or 3.

The permittivity and permeability can similarly be taken as that of either region with the respective ratios of these quantities again presumed to be less than an order of magnitude. Thus, the system is first EQS as the frequency is raised if the characteristic dimension, $a, b$ or $L$, is small compared to $l^*$, where the latter is based on the conductivity of the rod and the permittivity and permeability of either region. In the case where the charge relaxation time is the longest of the characteristic times, the EQS case, the magnetic induction is not important as the frequency is raised to the point where the sources begin to alter their distribution. In this case, the dominant source is the charge density, specifically the surface charge density. With each half-cycle, the surface charge density on the surface of the rod undergoes a sign reversal. To change this charge, the current density of (5) must be revised so that there is a component normal to the interface. In the "distributed circuit" picture of Fig. P15.3.2a, this is the current required to charge the capacitors. (In the next problem, the energy stored in the capacitors is used as a means of establishing the equivalent capacitance needed to account for the charging of the surface.)

In the case where the characteristic length is large compared to $l^*$, the system is MQS. The displacement current is negligible. This is equivalent to saying that the accumulation of charge has essentially no effect on the current density, which is itself solenoidal. Thus, the conductivity of the rod is large enough that the current that enters at one end is negligibly diverted by supplying surface charge, essentially all reaching the far end. However, because the magnetic induction is important, these currents try to link as little magnetic flux as possible. As suggested by the distributed circuit picture of Fig. P15.3.2b, the current distribution tends to crowd to the outer surface of the rod. The inductive reactance for a current circulating through the interior of the rod is less than that of a current nearer the surface. Thus, as the frequency is raised, the dominant field source, the current density, displays skin effect.

In Cartesian rather than cylindrical geometry, Example 10.7.1 illustrates the distribution of magnetic field and current density. The radial direction in this problem plays the role of the $x$ direction in the example. In both cases, the field and current density are independent of the axial direction ($y$ in the example and $z$ in this problem). One dimensional magnetic diffusion was pictured in Sec. 14.8 in terms of an $L-G$ transmission line (negligible capacitance). Note that this is equivalent to the $R-L$ distributed circuit used to schematically portray the MQS behavior in Fig. P15.3.2b. The transmission line would be an exact representation if the rod were replaced by a "slab" conductor and the return conductors were planar rather than circular cylindrical. Such a configuration is shown in Fig. S15.3.2b.

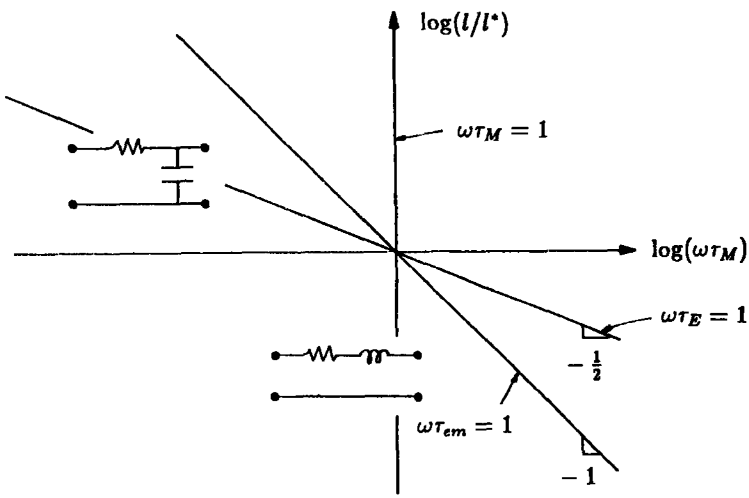
Demonsatration 10.7.1 makes use of a transformer rather than a current source to drive the currents through the conductor. In the limit where the probed conductor is very long compared to its depth, it gives rise to the same current distribution as obtained in the slab conductor of Fig. S15.3.2b. In the problem, the current distribution is somewhat different from that in the slab when the skin depth is on the order of the rod radius because of the cylindrical geometry.

(c) The conditions are as discussed in Sec. 14.9. So that the skin depth is large compared to the rod radius, the frequency must be low enough that the current distribution in the center conductor is essentially uniform. The inductance will nevertheless be self-consistently retained in the model provided that the conditions found in Prob. 14.9.2 are satisfied.

$$1 \ll ln(a/b) \tag{1}$$

(Here, the outer conductor has been effectively made to have an infinite conductivity by setting $\Delta \rightarrow \infty$ in the solution to Prob. 14.9.2.). Once we have decided to consider systems that are long in the axial direction, $z$, compared to the transverse dimensions and taken the quasi-one-dimensional model as representing the dynamics, it is interesting to see how the length, $l$, in the $z$ direction determines the order of the characteristic times

$$\tau_M = \frac{L}{R}; \quad \tau_{em} = \frac{l}{c} = l\sqrt{LC}; \quad \tau_E = l^2 RC \tag{2}$$




(c)

**Figure S15.3.2c**

In the limit where the inductance is not important, the system is a charge diffusion line as discussed in Sec. 14.9. Interestingly, the characteristic time associated with this EQS limiting model depends on the square of the length. Again, by contrast with a system having a single typical length, the interaction between the inductance and the resistance is independent of length (magnetic relaxation rather than diffusion). Thus, in constructing a length-frequency plane for sorting out the physical possibilities, it is the time $L/R$ that can be selected for normalizing the frequency. Thus, in this plane the critical lines are

$$\omega\tau_M = 1; \quad \omega\tau_{em} = 1 \Rightarrow \frac{l}{l^*} \equiv (\omega\tau_M)^{-1}; \quad \omega\tau_E = 1 \Rightarrow \frac{l}{l^*} = (\omega\tau_M)^{-1/2} \tag{3}$$

**and it follows (see Fig. S15.3.2c) that for the system to first be EQS as the frequency is raised, $l > l^* \equiv \sqrt{L/C}/R$.**

## 15.4 ENERGY, POWER, AND FORCE

**15.4.1** The electric field intensity in the three regions follows from Example 7.5.2. From (7.5.7) and (7.5.11), respectively,

$$\mathbf{E}^b = (v/L)\mathbf{i}_z \tag{1}$$

$$\mathbf{E}^a = -\frac{v}{ln(a/b)}\Big[\frac{z}{rL}\mathbf{i}_r + \frac{ln(r/a)}{L}\mathbf{i}_z\Big] \tag{2}$$

The magnetic field intensity is summarized in Example 11.3.1. From (11.3.10) and (11.2.12), respectively,

$$\mathbf{H}^b = \frac{\sigma v}{2L} r\mathbf{i}_\phi \tag{3}$$

$$\mathbf{H}^a = \frac{\sigma v}{L}\frac{b^2}{2r}\mathbf{i}_\phi \tag{4}$$

The required electric energy, magnetic energy, and dissipation follow by carrying out the piece-wise volume integrations.

$$w_e = \int_{-L}^{0}\int_{b}^{a}\frac{1}{2}\epsilon_a\mathbf{E}^a\cdot\mathbf{E}^a 2\pi r dr dz + \int_{-L}^{0}\int_{0}^{b}\frac{1}{2}\epsilon_b\mathbf{E}^b\cdot\mathbf{E}^b 2\pi r dr dz \tag{5}$$

$$w_m = \int_{-L}^{0}\int_{b}^{a}\frac{1}{2}\mu_a\mathbf{H}^a\cdot\mathbf{H}^a 2\pi r dr + \int_{-L}^{0}\int_{0}^{b}\frac{1}{2}\mu_b\mathbf{H}^b\cdot\mathbf{H}^b 2\pi r dr dz \tag{6}$$

and

$$p_d = \int_{-L}^{0}\int_{0}^{b}\sigma\mathbf{E}^b\cdot\mathbf{E}^b 2\pi r dr dz \tag{7}$$

Note that this last integral is essentially one of the two carried out in (5). Evaluation of these expressions, using (1)-(6), gives

$$\begin{aligned} w_e = {} & \frac{1}{2}v^2\Big\{\frac{2\pi\epsilon_a}{Lln^2(a/b)}\big[\frac{1}{3}L^2 ln(a/b) + \frac{a^2}{2}\{\frac{1}{2} + (b/a)^2[ln(b/a) - ln^2(b/a) - \frac{1}{2}]\}\big]\Big\} \\ & + \frac{1}{2}v^2\big[\epsilon_b\frac{\pi b^2}{L}\big] \end{aligned} \tag{8}$$

$$w_m = \frac{\pi\mu_a b^4 v^2\sigma^2}{4L}ln(a/b) + \frac{\pi\mu_b\sigma^2}{16L}v^2 b^4 \tag{9}$$

$$p_d = v^2\big[\frac{\sigma\pi b^2}{L}\big] \tag{10}$$

Written with the voltage replaced by the total current,

$$i = v\left(\frac{\sigma\pi b^2}{L}\right) \tag{11}$$

the magnetic energy, (9), becomes

$$w_m = \frac{1}{2}\left[\frac{\mu_a L ln(a/b)}{2\pi} + \frac{\mu_b L}{\pi 8}\right] i^2 \tag{12}$$

From a comparison of (8), (12), and (10), respectively, to

$$w_e = \frac{1}{2}Cv^2; \quad w_m = \frac{1}{2}Li^2; \quad p_d = i^2 R = v^2 G \tag{13}$$

it follows that the quasi-stationary parameters that model the system at frequencies that are low compared to either $R/L$ or $1/RC$, whichever is the lower, are

$$\begin{aligned} C = {} & \frac{2\pi\epsilon_a}{L ln^2(a/b)}\left\{\frac{1}{3}L^2 ln(a/b) + \frac{a^2}{2}\left[\frac{1}{2} + (b/a)^2\left[ln(b/a) - ln^2(b/a) - \frac{1}{2}\right]\right]\right\} \\ & + \epsilon_b \frac{\pi b^2}{L} \end{aligned} \tag{14}$$

$$L = \frac{1}{2}\left[\mu_a \frac{L ln(a/b)}{2\pi} + \frac{\mu_b L}{8\pi}\right] \tag{15}$$

$$G = \frac{\sigma_b \pi b^2}{L} \tag{16}$$

(Note that $L$ on the right is the length $L$ of the device, to be distinguished from the inductance $L$ on the left in (15).) Written in the form of (15.2.8), the ratio of the total magnetic to the total electric energy is, from (9) and (8)

$$\frac{w_m}{w_e} = K\left(\frac{b}{l^*}\right)^2; \qquad l^* \equiv \sqrt{\frac{\epsilon_a}{\mu_a}\frac{1}{\sigma}} \tag{17}$$

where

$$\begin{aligned} K \equiv {} & \left(ln(a/b) + \frac{\mu_b}{4\mu_a}\right) / \left\{\frac{4(a/b)^2}{ln^2(a/b)}\left[\frac{1}{3}(L/a)^2 ln(a/b)\right.\right. \\ & \left. + \frac{1}{2}\left[\frac{1}{2} + (b/a)^2\left[ln(b/a) - ln^2(b/a) - \frac{1}{2}\right]\right]\right] \\ & \left. + \frac{2\epsilon_b}{\epsilon_a}\right\} \end{aligned} \tag{17}$$

Provided the ratio of all dimensions and of the permittivities and permeabilities are on the same order, the coefficient $K$ is "of the order of unity."